

## N O T I C E

THIS DOCUMENT HAS BEEN REPRODUCED FROM  
MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT  
CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED  
IN THE INTEREST OF MAKING AVAILABLE AS MUCH  
INFORMATION AS POSSIBLE

(NASA-CR-160962) SPS FLEXIBLE SYSTEM  
CONTROL ASSESSMENT ANALYSIS Final Report  
(Rensselaer Polytechnic Inst., Troy, N. Y.)  
163 p HC A08/MF A01

M81-25122

CSCI 22B

Unclass  
26538

G3/15

FINAL REPORT

SPS FLEXIBLE SYSTEM CONTROL ASSESSMENT ANALYSIS

NASA CONTRACT: NAS 9-16053

*NASACR-160962*

COMPLETION DATE: March 15, 1981



Rensselaer Polytechnic Institute  
Troy, New York 12181



PRINCIPAL INVESTIGATOR:

Mark J. Balas  
Associate Professor of Systems Engineering  
Electrical, Computer and Systems Eng. Dept.  
Rensselaer Polytechnic Institute  
Troy, New York 12181

FINAL REPORT

TITLE: SPS FLEXIBLE SYSTEM CONTROL ASSESSMENT ANALYSIS

PRINCIPAL            Mark J. Balas  
INVESTIGATOR:      Associate Professor of Systems Engineering  
                     Electrical, Computer and Systems Eng. Dept.  
                     Rensselaer Polytechnic Institute  
                     Troy, NY 12181

NASA CONTRACT:    NAS 9-16053

COMPLETION DATE:   March 15, 1981

ORIGINAL PAGE IS  
OF POOR QUALITY

# ABSTRACT

This report addresses research into active (feedback) control of the Satellite Power System (SPS), a large mechanically flexible aerospace structure. The basic objective of this analysis is to develop an overall assessment of the interaction of the SPS structure and its active control systems.

Executive Summary: NAS-9-16053

In order to take maximum advantage of the space environment, especially the available solar energy, future spacecraft and satellites will tend to be large (0.1 - 100 kilometers in their largest dimensions) and very mechanically flexible structures. It is likely that such large space structures (LSS) will exhibit the following characteristics:

- (a) high total mass but low mass density (mass distributed throughout the structure over a large area);
- (b) a very large number of low frequency resonances (many below 1 hz);
- (c) very low natural damping (hence, easily excited resonances)

Because of their high level of flexibility, they will tend to react to forces, such as thermal and gravity gradients, solar pressure, and small impulse loadings, in ways which were heretofore ignorable in smaller structures. In particular, a concept like the Satellite Power System (SPS) will be especially vulnerable to these disturbance forces and yet, simultaneously, be expected to meet stringent performance requirements for shape fidelity, orientation alignment, vibration suppression, and pointing accuracy; however, all LSS will face similar difficulties.

Active control must be used to augment such structures in order to meet these specifications. The basic element of active control is feedback: sensors at various critical locations on the structure produce electronic signals proportional to local disturbance-induced errors in position, velocity, or acceleration; these signals are processed by control algorithms implemented in an on-line digital computer; the computer generates new electronic signals which are control commands applied to control

actuators such as thrust engines or torquing devices; the control actuators, placed at various locations on the structure, exert forces to bring the structure back into line and reduce the effects of any disturbances. Although the placement of appropriate actuators and sensors is an important item, the control algorithm, implemented in real-time by the on-line computer, is the principle component in this feedback link from sensors to actuators; it must be carefully designed to carry out its complex tasks within the physical limitations of the actuators and sensors and the computational limitations of the computer.

The essential element of the control algorithm is a model of the structural dynamics. This model is used to predict the behavior of the structure and compare it with the actual sensor measurements. It uses any errors it detects to improve its perception of the instantaneous structural behavior and generate control commands to cause this behavior to meet the system specifications, e.g. to keep the SPS microwave system accurately pointed at the Earth. For the rather rigid structures which comprise current spacecraft and satellites, such models are relatively simple; however, LSS require very complicated sets of partial differential equations and corresponding boundary conditions to predict the behavior at every location on the structure. A LSS is an example of what is called a distributed parameter system (DPS) which in theory must be described by an infinite number of modes of vibration, but even in practice a very large number of these modes is required to adequately approximate the dynamical behavior of the structure.

The number of modes included in the structure model is related to the computer memory capacity and its access time, i.e. how fast the computer

can calculate. If only a few modes are included in the controller's model, the computer can do the calculations very fast and there is imperceptible delay in the feedback system. However, if a large number of modes is included in order to accurately model the structural behavior, the computer must respond more slowly and the actuator feedback commands are delayed and arrive too late to correctly reduce the disturbance effects. In some cases, they arrive so late that they increase the disturbance effects and cause the system to be unstable. This dilemma of how many modes to keep (and which ones to throw out) and still be able to compute control commands rapidly is called model reduction; it is the crux of the LSS active control problem. Even if all the low frequency modes of a structure like SPS were retained in the controller model, they would more than exhaust the available computer capacity. Consequently, a model reduction must be performed and a trade-off made: Some modes are retained and others (thought to be less critical for performance) are left out of the controller's model.

Now we come to the controller-structure interaction problem. By necessity, we are forced to design a controller which actively controls the structure using a reduced model. The controller computer is only aware of the modes included in its model of the structure. The residual (i.e. unmodeled) modes are an intrinsic part of the actual structure, but the controller has no internal knowledge of them. When a particular sensor detects a local displacement error, this error is made up of contributions from all modes - modeled and residual; the part of the sensor output caused by residual modes is called observation spillover. The control computer accepts the sensor signal and compares it with its reduced internal

model to produce control commands. Such a command causes a force to be applied at an actuator; this force excites all modes - modeled or residual. The excitation of the residual modes is called control spillover (control energy "spills over" into the residual modes).

Consequently, through the structure controller we have feedback around the modeled modes of the structure (which we want) and feedback around the unmodeled modes through the control and observation spillover terms. This latter is the controller-structure interaction and it must not be ignored in the design and evaluation of active large space structure control. It may sound foolishly obvious but an actively controlled structure is quite different from a passive structure; by electronic feedback, the active controller changes the structure's characteristics, and its response to disturbances. It is intended that this change will improve the performance of the structure; however, if the controller-structure interaction assessment is not considered as a fundamental part of any active control design for LSS such as SPS, the sought after performance improvements will be lost. In fact, in some cases, the controller interaction with the unmodeled modes can cause them to be unstable which would mean an eventual loss of control of the structure.

This report presents a "top-down" approach to the design and analysis of active control systems for large aerospace structures, like the SPS. The introduction in Section I describes the LSS control issues and the objectives of this study in greater depth and gives some current references on the general problem. Sections II and III describe the main research performed:



- (a) development of a mathematical framework for the LSS control problem;
- (b) development of methods to synthesize active structure controllers from distributed parameter or finite element models of the structure;
- (c) development of methods to analyze and assess the controller-structure interaction and its impact on the successful operation of actively controlled structures.

Section IV describes the application of this approach to a generic model of the SPS which emphasizes the most stringent SPS control requirement: accurately pointing the microwave power transmission sub-system. Section V proposes and assesses methods to compensate and reduce the deleterious effects of the controller-structure interaction; this discussion includes structure, as well as controller, modifications. Section VI discusses an important issue for actively controlled structures: the control is only as good as the model. We have already seen the effects of the unmodeled modes; however, even the modeled modes may be rather poorly described due to our lack of knowledge of the zero gravity behavior of large structures. Consequently, some of the parameters of the structure may be poorly modeled; hence, there is a need for on-board estimation of structural parameters and the ability of the control algorithm to make use of these parameter estimates to adapt (i.e. tune) itself to the structure it is controlling. Sections VII and VIII present discussions and conclusions of the results of this study and recommendations for future research directions both in systems theory and experiment design. The papers published under this contract appear as Appendices A-E and G; Appendix F contains a paper which

was not supported by this contract but is germane to the topics discussed  
in Section VI.

## TABLE OF CONTENTS

	Page
I. Introduction to the SPS Active Control.....	1
II. Active Structure Control: Controller Design and Interaction Analysis.....	8
III. Active Structure Control Using Singular Perturbations....	12
IV. Controller-Structure Interaction For SPS: Application to A Generic Model.....	14
V. Compensation of the Controller-Structure Interaction.....	17
1.0 Compensation by Structural Modification.....	17
2.0 Compensation by Controller Modification.....	19
3.0 Special Purpose (Localized) Controllers.....	19
VI. Parameter Estimation and Adaptive Control of Structures..	21
VII. Discussion and Conclusions.....	23
VIII. Recommendations For Future Research in Large Aerospace Structures.....	25
Appendix	
A: Feedback Control of Dissipative Hyperbolic Distributed Parameter Systems with Finite Dimensional Controllers	
B: Finite Element Models and Feedback Control of Flexible Aerospace Structures	
C: Reduced-Order Feedback Control of Distributed Param- eter Systems Via Singular Perturbation Methods	
D: Stability of Distributed Parameter Systems with Finite Dimensional Compensators Via Singular Perturbation Techniques	
E: Closed-Loop Stability of Large Space Structures Via Singular and Regular Perturbation Techniques	
F: Toward Adaptive Control of Large Space Structures (with C. R. Johnson, Jr.)	
G: Reduced-Order Adaptive Controller Studies (with C. R. Johnson, Jr.)	

## I. INTRODUCTION TO THE SPS ACTIVE CONTROL

### 1.0 Introduction

The Satellite Power System (SPS), as described in [1], is an extremely large aerospace structure; its operation will require integration of structural design and active control systems to generate and transmit, via microwaves, large quantities of electrical power from geosynchronous orbit to receiving antennae on the earth's surface. The solar collector portion of the satellite is the largest piece of the structure, see [1] Figure 4; it has a low structural mass (in proportion to the total satellite mass) and is mechanically flexible with many low frequency resonances. In addition, the microwave transmitting antenna, although much smaller in size, is a flexible substructure which must be fine-pointed at the earth with minimal structural dynamic interaction with the solar array.

The following areas are most likely to require some active control to augment the structural design during in-space operation:

- (a) attitude control for solar orientation of the array structure;
- (b) shape or figure control of the array structure to maintain the nominal form;
- (c) fine earth pointing control of the microwave transmission antenna.

Active (or feedback) control would be implemented by on-line digital processing of signals, from a variety of sensors located on the structure, to produce control actuator commands, for thrust and momentum devices, to make the necessary corrections to achieve the above control objectives. These control tasks might be achieved with separate, or "decentralized", controllers or by a central control processor or some combination of these; the tasks would have to be accomplished simultaneously on the SPS.

## 2.0 Technical Discussion

### 2.1 Problem Statement

Since the SPS is a mechanically flexible structure, its dynamic behaviour is modeled by a continuum description, i.e. a set of coupled partial differential equations which model the flexible substructures, e.g. beams, plates, membranes, and a set of boundary conditions which model the interconnection of these substructures to each other and to rigid masses in the total SPS structure. This means that SPS is in the category of distributed parameter systems (DPS), i.e. those described by partial differential equations as distinguished from lumped parameter systems which are described by ordinary differential equations.

Such a DPS description of SPS would have the form:

$$m u_{tt} + D_0 u_t + A_0 u = F \quad (2.1)$$

where  $u(x,t)$  represents the (vector of) displacements (rotational and translational) from the equilibrium position of the structure  $\Omega$  at the point  $x$  and time  $t$  due to the external force distribution  $F(x,t)$ . The structural internal restoring forces  $A_0 u$  and damping  $D_0 u_t$  are generated by partial differential operators  $A_0, D_0$  with appropriate boundary conditions on  $u$ , e.g.

$$A_1 u + A_2 u_t = 0, \text{ on boundary of } \Omega \quad (2.2)$$

The mass distribution of the structure is given by  $m(x)$ . The force distribution is composed of disturbance forces  $F_D$  and control forces  $F_C$ :

$$F(x,t) = F_D(x,t) + F_C(x,t) \quad (2.3)$$

Control may be introduced via  $M$  force or momentum actuators with amplitudes  $f_i(t)$ :

$$F_C(x,t) = \sum_{i=1}^M b_i(x) f_i(t) \quad (2.4)$$

where  $b_i(x)$  are the influence functions of the individual actuators and

$f(t)$  is the vector of actuator commands  $f_i(t)$ . Observation of the system is done via  $P$  sensors:

$$y(t) = C_1 u + C_2 \dot{u} \quad (2.5)$$

where  $\dim y(t) = P$  and (2.5) allows for position, velocity, acceleration, or combinations of these measurements. There would also be sensor and actuator dynamics associated with (2.4)-(2.5) but these dynamics have not been shown; they would be part of the assessment.

The performance desired of the SPS in terms of pointing accuracy, shape control, and solar orientation may be stated in terms of closed-loop transient and steady-state behavior or as a performance function, such as modal energy or pointing error, to be minimized. Both types of performance specification will be taken into account in this assessment.

From DPS theory, the dynamical behavior of the uncontrolled SPS structure is described by an infinite number of vibration modes - the rigid body (zero frequency) modes plus all the independent (resonant frequency) flex modes. In practice, finite element structural analysis techniques are used to produce approximate modal data (mode frequencies and mode shapes) which yields a large-scale, lumped parameter model of the SPS (i.e. a large but not infinite number of modes). This large scale model suffers from model error (i.e. error in the modal data) and unmodeled residual modes of the SPS.

In addition, although the large scale model can be used for analysis and computer simulation purposes, it is usually too large to use for an on-board, real-time controller. The limited capacity of such controllers will require that a reduced-order model of the SPS be produced; this entails selection of a smaller subset of modes (or an aggregation of modes)

that gives a reasonably faithful representation of the total system behavior. The active controller design would be based on this reduced-order model. Such a controller would take the form:

$$\begin{cases} \dot{f}(t) = H_{11} y(t) + H_{12} z(t) \\ \dot{z}(t) = H_{21} y(t) + H_{22} z(t) \end{cases} \quad (2.6)$$

where  $\dim z(t) = S < \infty$ . The controller dimension  $S$  and the parameters  $H_{11}$ ,  $H_{12}$ ,  $H_{21}$ ,  $H_{22}$  would be determined directly from the reduced-order model of the SPS. The size of  $S$  and the operations involved in (2.6) are directly related to the capacity of the on-board control computer (or subcomputer).

## 2.2 Fundamental Research Problems for Active Control of SPS

The most fundamental control research problem for the SPS is the interaction of any on-board active controller (2.6), based on a reduced-order model, with the actual SPS structure (2.1)-(2.5). This structure-controller interaction occurs because of modeling error, control-device dynamics, and spillover due to modes not included in the reduced-order model (for "spillover", see [2]-[3]); this interaction is the most crucial research issue to be considered in this analysis. Other important issues to be assessed are the following:

- (1) selection of appropriate structure vibration modes to control to achieve desired system performance;
- (2) the effect of residual (uncontrolled) modes on system performance;
- (3) the effect of modeling error (i.e. modal approximation) on system performance;
- (4) the use of appropriate compensation, if needed, to counteract residual mode and model error effects

(5) the effect of actuator and sensor dynamics on controller design and system performance;

(6) controller design with multiple distributed actuators and sensors;

(7) control device (actuator or sensor) location, quantity, and characteristics necessary to achieve the basic control objectives;

(8) examination of the trade-offs available in the basic control approaches, including structural design, passive dampers, direct output feedback, and modal control with compensation feedback and

(9) separation control of the SPS array structure from the microwave antenna to reduce the effect of vibrations in the main structure on the beam-pointing accuracy.

### 2.3 Approach

This analysis will use the flexible structure control theory of [2]-[6] to develop a general mathematical framework for the assessment of basic active control systems procedures for the SPS. The development of this framework will be the principle element of this assessment. It will be able to begin with a continuum description of SPS, or a finite element description, or even a simplified continuum model, such as a plate model, and follow through, in a "top-down" manner, to the basic controller logic and provide an assessment of the fundamental issues raised in Section 2.2. Particular emphasis will be given to making the framework broad enough to include new control trends, e.g. [7]-[9], and, yet, capable of yielding answers to the above issues and to the assessment of other approaches to (reduced-order) modeling, e.g. traveling-wave vs. modal descriptions of SPS. It will not be limited to the current SPS reference system [1].



This approach will emphasize:

- (a) generic problems of SPS control;
- (b) the basic design logic for controller synthesis;
- (c) the trade-offs inherent in the basic control approaches for the attitude, figure, and pointing control tasks;
- (d) the interrelationship and interaction of the overall structural design and its models with the active controller design and operation;
- (e) the distributed parameter nature of the modeling, control and interaction problems.

REFERENCES FOR SECTION I

1. Satellite Power System: Concept Development and Evaluation Program, U.S. Dept. of Energy and National Aeronautics and Space Administration, Reference System Report DOE/ER-0023, October 1978.
2. M. Balas, Modal Control of Certain Flexible Dynamic Systems, SIAM J. Contr. and Opt. 19(1978), 450-462.
3. M. Balas, Feedback Control of Flexible Systems, Trans. Autom. Contr., AC-23(1978), 673-679.
4. M. Balas, Direct Output Feedback Control of Large Space Structures, J. of Astronautical Sciences, 27(1979), 157-180.
5. M. Balas, Direct Velocity Feedback Control of Large Space Structures, J. Guidance and Control, 2(1979), 252-253.
6. M. Balas, Reduced-Order Control of Large Structures in Space, presented at 17th Aerospace Sciences Meeting, New Orleans, LA, 1979.
7. M. Balas, Some Trends in Large Space Structure Control Theory: Fondest Hopes; Wildest Dreams, Invited Survey for Joint Automatic Control Conference, Denver, CO, 1979, to appear in revised form in IEEE Trans. on Autom. Control.
8. S. Seltzer, ed., Special Issue on Dynamics and Control of Large Space Structures, J. Astronautical Sciences, 27(1979), 93-214.
9. L. Meirovitch, ed., Proc. of 1st and 2nd Symposia on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, 1977 and 1979.

## II. ACTIVE STRUCTURE CONTROL: CONTROLLER DESIGN AND INTERACTION ANALYSIS

Our basic objectives in this study are to develop fundamental controller synthesis approaches for SPS structure control and to assess the interaction of the SPS structure with its active controller(s). An implementable active structure controller is one which can make use of a wide variety of possible sensors and actuators to synthesize on-line, via digital computer(s), the feedback control commands necessary to maintain the structural shape, attitude, and/or pointing requirements in the presence of disturbances. Such a controller must be finite-dimensional. The controller dimension must not be too high because this dimension corresponds to the on-line computer capacity (i.e., the size of the memory and the speed of memory access).

The fundamental issue is the finite-dimensional, implementable control of an infinite-dimensional, distributed parameter structure. Any controller synthesis technique must produce finite-dimensional controllers for the distributed parameter structure; this will always involve some kind of model reduction of the infinite dimensional system. However, due to this model reduction, there will always be a controller-structure interaction with residual (unmodeled) structure dynamics. In general, the sources of controller-structure interaction are the following:

- (a) structure modeling error (i.e., poorly known model parameters)
- (b) unmodeled control device dynamics
- (c) spillover due to structure dynamics unknown to the controller.

Therefore, it is not enough to produce active controller synthesis procedures for structural control; the controller-structure interaction

must be analyzed for any such procedure. This analysis is an intrinsic part of the problem and should be done with a view toward controller re-design for reduced interactions. Without a general mathematical framework which takes the above issues into account, it is impossible to handle the active structure control problem in any but a trial and error fashion.

In Paper A: Feedback Control of Dissipative Hyperbolic Distributed Parameter Systems with Finite Dimensional Controllers (presented at the 1979 IEEE Control and Decision Conference; revised and expanded version submitted to SIAM Journal of Control and Optimization) we have presented such a general mathematical framework for model reduction, controller synthesis, and interaction analysis. This framework is used to assess modal control approaches to active structures. In modal control, it is assumed that the vibration mode frequencies and mode shapes of the structure are known. A group of critical modes are chosen, e.g., by their excitability by the disturbance frequency spectrum or by their contribution to a performance index such as the microwave antenna pointing accuracy; the choice of critical modes is not a trivial issue by any means and merits further investigation. These critical modes form the reduced-order model of the structure. The modal controller is synthesized with only the critical mode data. In Paper A, the most general linear controller is considered; this allows trade-offs in the controller design, the number and location of control devices (actuators and sensors), and the on-line computer capacity. Very simple modal criteria are developed to assist the designer in assessing the available design trade-offs. The controller-structure interaction is analyzed, and bounds on spillover and model error are produced; these bounds indicate

the tolerable levels of these interaction terms for stable operation of the active structure. When the bounds are satisfied it makes sense to use the modal controller as designed; when they are not satisfied, the designed controller should be used with caution and, more reasonably, it should be redesigned or spillover compensation should be added. We will say more later about spillover compensation. The mathematical framework presented in Paper A is very general and can handle many other types of active structure control; it is not limited to modal control.

The most common tool available to structural dynamicists is NASTRN which is a large-scale computer code based on a finite element analysis of a structure. Approximate mode shapes and frequencies are available from NASTRN and can be used in active controller synthesis. The fact that the modal data is approximate adds the effect of model error to that of spillover to further complicate the controller-structure interaction. In Paper B: Finite Element Models and Feedback Control of Flexible Aerospace Structures, the structure control synthesis and analysis problem is considered in detail for finite element models of the structure. Bounds on tolerable model error and spillover are produced to indicate when an implementable active controller based on a consistent finite element approximation of a structure like SPS makes sense and can be expected to operate stably.

Among the practical constraints on structure control, we are inclined to assume that point actuators and sensors will be used; these may be placed at many locations on the structure. In some special situations a distributed actuator, e.g., use of electromagnetic field interaction of the SPS and the earth, may be available. Such distributed actuators and sensors are not excluded from the framework established in Paper A and may be used to advantage when they exist.

It should be mentioned that in all of our work on active structure control we are talking about feedback or closed-loop control of the structure. In "Control-Structure Interaction in a Free Beam" by S. Shrivastava and R. Ried (NASA Report JSC 16699, May 1980), an energy approach is used to study the controller-structure interaction for a free beam with on-off thruster control. The paper illustrates the excitability of structure modes to such nonlinear control forces. This is certainly an important consideration; however, one should be careful not to draw many detailed conclusions from such an analysis because the control is open-loop. Many of the controller-structure interaction effects appear because of the feedback nature of active structure control. The analysis of Shrivastava and Ried aids in understanding the complex way in which a flexible structure responds to open-loop thruster loads; it is quite complementary to our analysis which considers linear forces generated by feedback control systems.

The above mentioned papers A and B are Appendices A and B of this report.

ORIGINAL PAGE IS  
OF POOR QUALITY

### III. ACTIVE STRUCTURE CONTROL USING SINGULAR PERTURBATIONS

In the previous section, we presented a general framework for active structure control synthesis and analysis. It was based on a regular perturbation viewpoint, i.e., the spillover and model error terms were taken as perturbations of the desired system; this is how we arrived at our spillover and model error bounds for closed-loop stability. A different viewpoint which uses singular (as opposed to regular) perturbations is described and analyzed in Paper C: Reduced-Order Feedback Control of Distributed Parameter Systems Via Singular Perturbations (presented at the 1980 Princeton Conference on Information Sciences and Systems; revised version submitted to Journal of Mathematical Analysis and Applications) and Paper D: Stability of Distributed Parameter Systems with Finite Dimensional Compensators Via Singular Perturbation Methods (presented at 1980 International Congress on Applied Systems Research and Cybernetics). Applications of the general approaches to aerospace structures are discussed in Paper E: Closed-Loop Stability of Large Space Structures Via Singular and Regular Perturbation Techniques (presented at 1980 IEEE Control and Decision Conference).

The principal difficulty in applying the singular perturbation approach is to arrive at a physically meaningful singular formulation. When this can be done the system separates into a critical subsystem that must be controlled and a residual subsystem whose overall effect can be approximated statically. This makes the controller synthesis and analysis very different from that of Section II and we feel this justifies a separate section devoted to this approach.

No single approach to the active structure control problem can be

expected to yield all the answers. Each one has its special advantages and disadvantages in a given application. In this and the previous section, we have presented two very general viewpoints which incorporate the controller synthesis and the interaction analysis which we feel is crucial to the solution of this complex problem.

Papers C, D, and E are Appendices C, D, and E of this report.



#### IV. CONTROLLER-STRUCTURE INTERACTION FOR SPS: APPLICATION TO A GENERIC MODEL

Although the assessment approaches described in the previous sections may be applied to a wide variety of flexible structure models, including finite element models, we have chosen to look carefully at their application to a simple model of SPS and the (potentially) most difficult control problem: accurately pointing the microwave antennae on the flexible solar array structure. Of course, more detailed models will exhibit more aspects of this complex problem but we are operating on the trite-but-true principle: You have to start somewhere!

The generic model we have used for SPS is a flexible beam with lumped masses at both ends; this model was also considered in "Projected Models of Solar Power System" by J. Juang (Jet Propulsion Laboratory Engineering Memo 347-58, Feb. 1980).

The dynamics are given by:

$$\begin{cases} u_{tt} + D_0 u_t + A_0 u = B_0 f \\ y = C_0 u + E_0 u_t \end{cases} \quad (4.1)$$

where

$$A_0 u = \alpha u_{xxxx} - \beta u_{xx} \quad (4.2)$$

with appropriate boundary conditions and  $u(x, t)$  is the transverse displacement off equilibrium of the beam of length  $l$ . The damping operator  $D_0$  is chosen later; its form is not of great importance since it only adds a small amount of dissipation to the structure. We consider several actuator-sensor arrangements.

The performance index is given by:

$$J = \int_0^{\infty} [E^2(0, t) + E^2(l, t) + r^T R r] dt \quad (4.3)$$

where the pointing error  $E(x, t)$  is given by:

$$E(x, t) = \theta(x, t) - \theta_r(x) \quad (4.4)$$

where  $\theta_r(x)$  is the desired pointing direction at the location  $x$  and

$$\theta(x, t) = \frac{\partial \psi(x, t)}{\partial x}$$

is the angle of the normal (or pointing) vector at location  $x$  at time  $t$ . The control penalty term has a positive definite weighting matrix  $R$ ; this penalty term can be selected to keep the actuator commands within practical limits.

The system (4.1) together with the performance index (4.3) forms a distributed parameter optimal control problem; this can be brought into the form of an optimal regulator by considering:

$$e(x, t) = u(x, t) - u_r(x) \quad (4.5)$$

where  $u_r(x)$  is chosen so that:

$$\frac{\partial u_r}{\partial x} = \theta_r \quad (4.6)$$

It is easily shown that

$$\begin{cases} e_{tt} + D_0 e_t + A_0 e = E_0 f \\ y = C_0 e + E_0 e_t + C_0 u_r \end{cases} \quad (4.7)$$

when (as is the case by proper choice of  $u_r$ ):

$$\Lambda_0 u_r = 0 \quad (4.8)$$

we shall choose to measure

$$\tilde{y} = y - C_0 u_r \quad (4.9)$$

instead of  $y$ ; this can always be done because  $C_0 u_r$  is a known time-invariant term. Also, note that the pointing error becomes:

$$E(x, t) = \frac{\partial e}{\partial x} \quad (4.10)$$

Combining (4.5), (4.7), (4.9), (4.10), and (4.3), we obtain a distributed parameter optimal regulator problem, as desired. This problem may be analyzed using the methods of Paper A. We have developed active modal controllers by two straightforward methods:

- (a) produce the infinite dimensional controller (which cannot be implemented) and truncate it to obtain the finite dimensional controller;
- (b) truncate the distributed parameter system and performance index, and generate the finite dimensional controller from this reduced-order model.

The latter method seems the most natural from an engineering viewpoint. When the exact modes are known (as they are in this model), the two methods yield essentially the same controller.

This application and the study of its properties form the basis for the masters thesis of my research assistant, K. Mashayekhi; this thesis will be completed and available by June 1, 1981.

## V. COMPENSATION OF THE CONTROLLER-STRUCTURE INTERACTION

### 1.0 Compensation by Structural Modification

Some alleviation of the controller-structure interaction may be obtained by purely structural modification or redesign. An optimal overall structural design may be done to redistribute the mass and increase the stiffness. From our control point of view, we need increased damping without serious loss of stiffness; since the stiffness is mainly a function of configuration and the damping is a function of materials and construction, this may be attainable with structural reconfiguration and use of composite materials. Also, we need to know the system parameters as well as possible; perhaps, an attempt should be made to structurally isolate subsystems or to yield a structure composed of well-approximated subsystems. One thing is very clear, the best compromise of structural stiffness vs. damping would be made in the preliminary design phase; such decisions require a basic understanding of the effects of structural configuration and materials to meet the mission requirements.

Structural add-ons include coatings to increase high frequency damping and (passive or active) member dampers to increase low frequency damping. The latter are especially desirable because it is the low frequency damping that most needs augmentation; perhaps, some combination of dampers plus a more sophisticated active control can be used to achieve the structural control requirements.

Of course, much of the structural design is constrained by factors other than the ultimate function of the overall structure, e.g., transportability and construction in orbit. Nonetheless, in the more distant

future we would expect to see control-configured structures, i.e., structure and active control designed and optimized as one integrated package. Such an integrated optimization is not simply a matter of producing a "super computer program"; a much deeper understanding of structural behavior in the presence of active control will be required to generate control-configured aerospace structures.

## 2.0 Compensation by Controller Modification

In general, we feel that structural modification alone is quite limited as a means to reduce the controller-structure interaction. The interaction problem occurs because of the combination of mechanical structure and control electronics; this suggests further use of electronic compensation as an alternative.

Actually, there are many ways to introduce this spillover compensation. The basic requirements are the following:

- (a) compensation should be added-on to the existing controller
- (b) compensation should not substantially increase the number of control devices or the dimension of the controller (i.e., the on-line computer capacity).

The methods we have investigated are prefiltering, cancellation, residual aggregation, and enhancement. Each of these has its advantages and disadvantages; none is a panacea. The most natural seems to be prefiltering, i.e., filtering the sensor outputs to reduce the spillover frequency components; however, phase distortion due to the filter can have as deleterious an effect on the stability as the spillover it removes.

We feel this area has not received sufficient attention; there are many promising directions for electronic compensation methods which need further investigation.

## 3.0 Special Purpose (Localized) Controllers

Decentralized control in general sounds like a good bet for active structure control. It has been used successfully in the area of electric

power generation and distribution; this application area has some surprising theoretical similarities to flexible aerospace structures. Therefore, we do not want to discount the possible benefits of a decentralized approach to structure control; however, we also do not want to forget the extremely difficult technical problems that continue to impede the development of a general theory of decentralized control.

Still, a localized approach has a great deal of merit if we exploit properties of the particular application. Specifically, the SPS microwave pointing problem might be attacked in the following localized way: treat the flexible antenna as the precision system, i.e., the one in need of accurate control, and the solar array as a disturbance system acting upon the antenna; try to design a disturbance counteracting control to point the antenna and simultaneously reduce disturbances from the solar array structure. Such an approach was first considered in "An Active Modal Control System Philosophy for a Class of Large Space Structures," by M. Balas and J. Canavin (presented at 1977 Blacksburg Symposium).

Of course, a centralized controller with a complete dynamic model of the array structure with antennae attached can do a much better job than the above-described localized control. However, the centralized controller is much more complicated and may prove impractical to implement while the more simple localized controller could be implemented and may prove to be nearly as effective as the centralized scheme. The trade-offs in such localized structure control schemes remain to be more carefully examined.

ORIGINAL PAGE IS  
OF POOR QUALITY

## VI. PARAMETER ESTIMATION AND ADAPTIVE CONTROL OF STRUCTURES

Active control of structures is based on a complete knowledge of certain system parameters; for example, in modal control, the mode frequencies and mode shapes must be known (at least, the mode shapes must be known at the actuator-sensor locations). In practice, these parameters are poorly known and may vary with time. The control designer has two choices:

- (a) make the design robust, i.e., capable of carrying out the control task even though the parameters have substantial error;
- (b) make the design adaptive, i.e., use the sensor information to estimate poorly known parameters on-line and have the controller tune itself up.

It is difficult to produce robust controllers for highly accurate tasks without introducing unacceptably high gains; therefore, we have concentrated on the adaptive approach. However, one should certainly check the sensitivity of a given structure controller to parameter inaccuracies before proceeding with parameter estimation and/or adaptive control.

A general approach to parameter estimation and adaptive control for aerospace structures is presented in Paper F: Toward Adaptive Control of Large Structures in Space by M. Balas and C. R. Johnson, Jr. (from Applications of Adaptive Control, K. S. Narendra and R. Monopoli, eds., Academic Press, 1980). This paper was not written under this contract; however, we include it because it discusses the special problems of this field within the context of our general mathematical framework. In addition, some preliminary investigation (accomplished with NASA support



this year) on adaptive control of reduced-order systems is presented in Paper G: Reduced-Order Adaptive Controller Studies by C. R. Johnson, Jr., and M. Balas (presented at 1980 Joint Automatic Control Conference).

Papers F and G are Appendices F and G of this report.

## VII. DISCUSSION AND CONCLUSIONS

No one who has worked in the field of aerospace structures would argue that active control of a structure as large and unruly as the proposed Satellite Power System is anything but a monumentally difficult task. Furthermore, no one but a fool or a madman would undertake to solve the basic SPS control problems in a one-year study. Hoping not to be judged the former and unsure about the latter, we have nevertheless undertaken the study of some of these problems under this contract.

Throughout this work, our emphasis has been on the generic problems and the distributed parameter nature of SPS active structure control with the goal of obtaining a deeper understanding of structure control technology by focusing on an extremely complicated and difficult structure. We have gone in many directions during this study; often, in order to cover as much ground as possible, we have developed a given direction only to the point where it began to reveal its potential. Still, this work is not without substance. We have developed a mathematical framework for the assessment of the active control of aerospace structures like (although not restricted to) the SPS. From this we have generated quite general techniques to:

- (a) synthesize structure controllers from distributed parameter or finite element models of the structure;
- (b) analyze the controller-structure interaction on the closed-loop stability;
- (c) compensate, by structure or controller modification, to reduce the interaction effects.

This has been done within the constraints of maximum use of existing control and structure knowledge and tools, allowance for a variety of SPS models and configurations, and ability to include new developments in control and/or structure technology.

We have accomplished our principal objectives in this study:

- (1) to develop basic active structure control approaches for SPS and assess the design trade-offs
- (2) to develop mathematical methods to assess the effects of controller-structure interaction on the closed-loop stability
- (3) to develop numerical methods to compute the stability bounds of (2).

This work has produced seven published papers and a masters thesis. We have presented various aspects of the work in several national conferences and in seminars at L. B. Johnson Space Center and the Jet Propulsion Laboratory throughout the year.

We hope our contributions and the perspective presented here will further the understanding of Satellite Power System control and active control of aerospace structures in general.

# VIII. RECOMMENDATIONS FOR FUTURE RESEARCH IN LARGE AEROSPACE STRUCTURES

The research reported here has been on the development of systems theory. It is in this area that we shall make our principal recommendations.

Two areas, out of many which we have touched on in this report, stand out as major issues in the successful development of active structure control:

- (1) Structural modeling for active control, i.e., modeling of the structure explicitly for feedback control purposes as opposed to assessment of dynamical behavior under external loads;
- (2) On-line parameter estimation and adaptive control.

The first of these really is an indication of the different viewpoints of the structural dynamicist and the structural control designer; emphasis on this problem would go a long way toward a much-needed rapprochement of these two disciplines. One cannot realistically expect to operate large aerospace structures without some resolution of the second issue; yet, on-line parameter estimation and adaptive control techniques lead to some of the most complex theoretical issues in distributed parameter systems.

There is a clear need to look beyond the systems theory aspects of large aerospace structures and to begin to consider the development of ground-based, and eventually in-space, experiments in actively controlled structures. Of course, no successful ground-based experiment can begin to approach the dimensions of something like the SPS; however, this is not necessary to gain the basic understanding that is needed to cope with

this new area of technology.

Instead, we may begin to investigate particular structural behavior. For example, ground-based experiments to determine the dynamical behavior of flexible membranes attached to semi-rigid structures would help us to understand a major component of the SPS: the interaction of the solar-cell blankets with their supporting structure. Furthermore, active suspension control could be introduced into such an experiment to investigate its capacity to suppress the blanket vibrations without seriously increasing vibrations in the supporting structure - a clear application of the systems concepts developed in this report to design and assess the controller-structure interactions.

Other experiments in active control of beams and plates could be designed and carried out to assess the potential of active structures and to illuminate the problems. Although computer simulation is an excellent tool for analysis of structural behavior, it is only as good as the models of the phenomena to be studied. These models require substantial improvement in order to successfully predict the complex behavior of large aerospace structures. Such improvements can only grow out of a better understanding of structures and their interaction with control - an understanding which can be greatly aided by laboratory, and space-shuttle-based, experiments with generic structures.

## ACKNOWLEDGEMENTS

We wish to acknowledge the cooperation and encouragement of Dr. R. C. Ried as well as the many helpful discussions with him and others at the L. B. Johnson Space Center during the course of this contract.

We would especially like to thank Dr. F. Stebbins for supplying much helpful information on the SPS structural dynamics.

APPENDIX A-G

# A. FEEDBACK CONTROL OF DISSIPATIVE HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS WITH FINITE DIMENSIONAL CONTROLLERS

Mark J. Balas  
Electrical and Systems Engineering Department  
Rensselaer Polytechnic Institute  
Troy, New York 12181

## ABSTRACT

The class of dissipative hyperbolic distributed parameter systems is defined here; it includes many physical phenomena such as mechanically flexible structures and certain linear wave propagation problems. Feedback control theory for this class of systems is investigated. Although some results on infinite dimensional controllers are presented, the emphasis is on practical (i.e., finite dimensional) controllers based on reduced-order models of the distributed parameter system and the closed-loop operation of such controllers in the actual system. Sufficient conditions, in terms of "spillover" bounds for the unmodeled residuals, are presented for exponentially stable closed-loop operation.

## 1.0 INTRODUCTION

Many physical phenomena are best described by partial differential equations, e.g., [1], Chapt. 1, and hence, are distributed parameter systems (DPS). Several books have been written on DPS and control, e.g., [2]-[7], as well as, a large number of papers, e.g., the surveys [8]-[9]; applications of DPS control range from mechanically flexible structures to wildfire suppression, e.g., the survey [10]. Our special area of interest has been application to large flexible aerospace structures, e.g., the surveys [11]-[12];



although the results obtained in this paper may be applied to such structures, they can be applied to a much larger class of problems, namely, dissipative hyperbolic DPS.

## 2.0 DISTRIBUTED PARAMETER SYSTEM DESCRIPTION: DISSIPATIVE HYPERBOLIC SYSTEMS

We consider DPS described by the following:

$$\begin{cases} \frac{\partial v(t)}{\partial t} = A v(t) + B f(t), & v(0) = v_0 \\ y(t) = C v(t) \end{cases} \quad (2.1)$$

where the system state  $v(t)$  is in a Hilbert space  $H$  with inner product  $(\cdot, \cdot)$  and associated norm  $\|\cdot\|$ . The differential operator  $A$  is linear, time-invariant, has domain  $D(A)$  dense in  $H$ , and generates a  $C_0$  semigroup  $U(t)$  on  $H$ . The DPS is called dissipative hyperbolic here when the semigroup  $U(t)$  has the following "growth" property:

$$\|U(t)\| \leq M_0 e^{-\varepsilon t}, \quad t \geq 0 \quad (2.2)$$

where  $\varepsilon \geq 0$  and  $M_0 \geq 1$ ; when  $\varepsilon > 0$ , we say the DPS is strictly dissipative. If  $\varepsilon = 0$ , we assume  $M_0 = 1$ . For more details on semigroups, see [6]-[7], [13]-[14].

The control is introduced via  $M$  inputs:

$$B f(t) = \sum_{i=1}^M b_i f_i(t) \quad (2.3)$$

ORIGINAL PAGE IS  
OF POOR QUALITY

where the actuator influence functions  $b_i$  are in  $H$ . The system is observed via  $P$  sensors whose outputs  $y_j(t)$  form the vector  $y(t) = [y_1(t), \dots, y_P(t)]^T$

and are given by

$$y_j(t) = (c_j, v(t)) \quad (2.4)$$

where the sensor influence functions  $c_j$  are in  $H$ . The rank of the input operator  $B$  is  $M$  and that of the output operator  $C$  is  $P$ . In most DPS applications the control and observation must be done with a finite number of devices; hence, these restrictions on  $B$  and  $C$  are natural. Usually, the control devices will be localized, i.e., the influence functions  $b_i$  and  $c_j$  will be nearly Dirac delta functions.

The class of dissipative hyperbolic systems includes mechanically flexible structures, where  $\epsilon$  is related to the natural or materials damping in the structure, and certain wave propagation problems which are described by symmetric hyperbolic systems of partial differential equations, e.g., [15]; in fact, it includes most oscillatory DPS. It does not include (so called) parabolic problems like the heat equation; for these systems, the semigroup  $U(t)$  is holomorphic [16] and the control problems are somewhat different. Control of parabolic DPS has been considered in, for example, [6], [16]-[19].

Our control approach to dissipative hyperbolic DPS emphasizes implementability, i.e., from the sensor outputs  $y(t)$  in (2.4), the controller must synthesize control commands  $f(t)$  in (2.3) to adequately stabilize and improve the performance of the DPS, and these commands must be synthesized with an on-line computer. Consequently, the control synthesizer must be a finite dimensional dynamical system (usually of low dimension). This places reasonable, practical constraints on the DPS control problem [20].

Infinite dimensional controller results are presented in Sec. 3, reduced order modeling and implementable controllers are discussed in Secs. 4 and 5. Our main results on closed-loop stability are presented in Sec. 6 with

application of these concepts to control of large flexible structures in  
Sec. 7 and general conclusions in Sec. 8.

### 3.0 INFINITE DIMENSIONAL CONTROLLER

Although the results presented in this section may not be implementable (and, in many applications, they are not), they do give some insight into the DPS control problem.

Control laws for DPS, as in the case for lumped parameter systems, often take the form of gains multiplying full state feedback:

$$f^*(t) = G v(t) \quad (3.1)$$

where  $G$  is a bounded control gain operator; this is especially true when a performance index is optimized, e.g., [6], Chapt. 14. However the states  $v(t)$  for a DPS are in an infinite dimensional space and, in most cases, can not be recovered with a finite number of sensors as in (2.4). Consequently, one is led to a state estimator (again, as in the lumped parameter case) of the form:

$$\begin{cases} \frac{\partial \hat{v}(t)}{\partial t} = A \hat{v}(t) + B f(t) + K y(t) - \hat{y}(t), \hat{v}(0) = 0 \\ \hat{y}(t) = C \hat{v}(t) \end{cases} \quad (3.2)$$

where  $K$  is a bounded estimator gain operator constructed so that  $\hat{v}(t)$  remains in  $D(A)$ . The estimator error,  $e(t) \equiv \hat{v}(t) - v(t)$ , satisfies

$$\frac{\partial e(t)}{\partial t} = (A - K C) e(t), \quad e(0) = -v_0 \quad (3.3)$$

It is natural to take the control law

$$f(t) = G \hat{v}(t) = G v(t) + G e(t) \quad (3.4)$$

in place of (3.1) and hope for the best.

The following theorem gives sufficient conditions for the exponentially stable operation of the infinite dimensional controller (3.2) and

ORIGINAL PAGE IS  
OF POOR QUALITY

(3.4) in closed-loop with the DPS (2.1), (2.3), (2.4):

### THEOREM 3.1

If  $(A, B)$  and  $(A^*, C^*)$  are exponentially stabilizable, then there exist bounded operators  $G$  and  $K$  such that the infinite dimensional controller produces an exponentially stable closed-loop system and  $\hat{v}(t)$  converges exponentially to the DPS state  $v(t)$ .

The proof of this theorem uses results from [21] and appears in Appendix I; a DPS  $(A, B)$  of the form (2.1) is exponentially stabilizable if a bounded gain operator  $G$  exists which causes the semigroup  $U_1(t)$  generated by  $A + B G$  to satisfy

$$\|U_1(t)\| \leq M_1 e^{-\epsilon_1 t}, \quad t \geq 0 \quad (3.5)$$

where  $M_1 \geq 1$  and  $\epsilon_1 > 0$ .

It is known from [22] that, if  $\epsilon = 0$  in (2.2), then neither  $(A, B)$  nor  $(A^*, C^*)$  can be exponentially stabilizable because  $B$  and  $C^*$  are finite rank (and, hence, compact). This means that, if the open-loop DPS is not strictly dissipative (no matter how small  $\epsilon$  is), then no amount of controller feedback will make the closed-loop system exponentially stable.

The infinite dimensional controller may also be written in convolution form:

$$f(t) = \int_0^t \hat{G} \hat{U}(t - \tau) K y(\tau) d\tau \quad (3.6)$$

where  $\hat{U}(t)$  is the  $C_0$  semigroup generated by  $A + B G - K C$ . However, it may not be possible to implement either (3.6) or (3.2) and (3.4) with finite dimensional devices. Of course, either version of the infinite dimensional controller may be approximately implemented and this is the subject of the next section.

#### 4.0 REDUCED-ORDER MODELS OF DPS

Even though the underlying structure of a DPS may be described by (2.1)-(2.4) and, hence, is infinite dimensional, the controller may have to be based on a finite dimensional approximation of the original DPS, in order to be implementable in the sense of Section 2.0. Such an approximation or reduced-order model (ROM) is a projection (not necessarily orthogonal) of the DPS on  $H$  onto an appropriate finite dimensional subspace  $H_N$  of  $H$ . The ROM subspace  $H_N$  has dimension  $N$  and its projection is denoted by  $P_N$ ; the residual subspace  $H_R$  associated with  $H_N$  completes the decomposition  $H = H_N \oplus H_R$  and its projection is denoted by  $P_R$ . The total DPS state  $v$  is the sum of the ROM state  $v_N \equiv P_N v$  and the residual state  $v_R \equiv P_R v$ :

$$v(t) = v_N(t) + v_R(t) \quad (4.1)$$

The choice of subspace  $H_N$  and the projection  $P_N$  completely specifies the residual subspace  $H_R$  and its corresponding projection  $P_R$ ; this choice is usually dictated by the physical application and knowledge (or lack of knowledge) about the specific DPS involved. In many situations, the partial differential equations or the corresponding boundary conditions are too complicated to permit simple closed-form solutions. Consequently, an approximation of DPS, such as the Rayleigh-Ritz-Galerkin or finite element method, [23]-[24], must be used to deal with the problem numerically; this is one way to produce a ROM of the DPS but it is by no means the only way to generate such approximations.

We will restrict our attention in this paper to consideration of ROM's obtained from projection (not necessarily orthogonal) onto a pair of reducing subspaces  $H_N, H_R$  for the differential operator  $A$ , i.e., the following

conditions hold:

$$H = H_N \oplus H_R \quad (4.2a)$$

$$P_N (D(A)) \subseteq D(A) \quad (4.2b)$$

$$P_N A P_R v = 0 \quad (4.2c)$$

$$P_R A P_N v = 0 \quad (4.2d)$$

From [25], p. 172, these conditions are equivalent to the condition that

$P_N$  commutes with  $A$ :

$$v \in D(A) \rightarrow P_N v \in D(A) \quad (4.3a)$$

$$A P_N v = P_N A v \quad (4.3b)$$

Of course,  $P_R$  also commutes with  $A$ .

Throughout the rest of this paper, any ROM considered for (2.1) will be based on some pair of reducing subspaces  $H_N$  and  $H_R$  for  $A$ . The projection onto these subspaces may or may not be orthogonal; when it is orthogonal, we will have  $H_R = H_N^\perp$  and

$$\|v\|^2 = \|v_N\|^2 + \|v_R\|^2 \quad (4.4)$$

and

$$\|P_N\| = \|P_R\| = 1 \quad (4.5)$$

An example of reducing subspaces occurs when  $H_N$  is a modal subspace, i.e.,

$$H_N = \text{span} \{ \phi_1, \dots, \phi_N \}$$

where  $A \phi_k = \lambda_k \phi_k$  for eigenvalues  $\lambda_k$  with corresponding eigenfunctions  $\phi_k$ ; these eigenvalues  $\lambda_k$  must be separated from the rest of the spectrum of  $A$  (i.e., a rectifiable, simple closed curve can be drawn in the complex plane

so as to enclose an open set containing  $\lambda_1, \dots, \lambda_N$  in its interior and the rest of the spectrum of  $A$  in its exterior). From [25], p. 178,  $H_N$  and  $H_R$  form a pair of reducing subspaces for  $A$  and  $A_N \equiv P_N A P_N$  is a finite dimensional operator whose spectrum is exactly  $\{\lambda_1, \dots, \lambda_N\}$  while the spectrum of  $A_R \equiv P_R A P_R$  is the rest of the spectrum of  $A$ . Note that the projections are orthogonal when  $A$  is normal [25], p. 277.

We emphasize that other reducing subspaces may exist for  $A$  but the modal one is useful when a finite set of separable eigenvalues, and their corresponding eigenvectors, can be found; this is the case, for example, when  $A$  has compact resolvent ([25], p. 187). Also, the modal subspace is physically reasonable for producing ROMs of many DPS such as mechanically flexible structures. It should be noted that, when the exact modes are not immediately available (even though they may be known to exist) and approximate modes are used for  $H_N$ , the subspaces  $H_N$  and  $H_R$  are not necessarily reducing subspaces.

We have the following result for dissipative hyperbolic DPS:

#### THEOREM 4.1

ORIGINAL PAGE IS  
OF POOR QUALITY

If  $H_N$  and  $H_R$  are reducing subspaces for  $A$  in (2.1), then this DPS decomposes into

$$\dot{v}_N(t) = A_N v_N(t) + B_N f(t) \quad , \quad v_N(0) = P_N v_0 \quad (4.6)$$

$$\frac{\partial v_R(t)}{\partial t} = A_R v_R + B_R f \quad , \quad v_R(0) = P_R v_0 \quad (4.7)$$

$$y(t) = C_N v_N(t) + C_R v_R(t) \quad (4.8)$$

where  $v_N = P_N v$ ,  $v_R = P_R v$ ,  $A_N = P_N A P_N$ ,  $B_N = P_N B$ ,  $C_N = C P_N$ , etc.

In addition, the projections  $P_N$  and  $P_R$  commute with the semigroup

$U(t)$  generated by  $A$  and  $A_R$  generates the  $C_0$  semigroup  $U_R(t) \equiv P_R U(t) P_R$



with the growth property:

$$||U_R(t)|| \leq M_R e^{-\epsilon t}, \quad t \geq 0 \quad (4.9)$$

where  $M_R \equiv ||P_R|| M_0$  with  $(M_0, \epsilon)$  as in (2.2).

The proof of this theorem uses results in [25] and appears in Appendix II.

The ROM corresponding to the projection of (2.1) onto the reducing subspaces  $H_N$  and  $H_R$  is defined by (4.6) and (4.8) with  $C_R$  assumed zero:

$$\begin{cases} \dot{v}_N(t) = A_N v_N(t) + B_N f(t) & , \quad v_N(0) = P_N v_0 \\ \tilde{y}(t) = C_N v_N(t) \end{cases} \quad (4.10)$$

or (4.10) may be abbreviated as  $(A_N, B_N, C_N)$ . In particular, when  $H_N$  is the modal subspace for  $N$ (separated) eigenvalues of  $A$ , we call the corresponding ROM - the modal ROM.

The interaction terms  $B_R f$  and  $C_R v_R$  are called control and observation spillover, respectively. It is through these terms that a feedback controller can interact with the residual subsystem (4.7). This would not be the only interaction if the ROM had been based on nonreducing subspaces for  $A$ ; in that case model error terms -  $P_N A P_R v_R$  and  $P_R A P_N v_N$  - would have appeared in (4.6) and (4.7), respectively. However, these terms are zero when reducing subspaces are used for the ROM (as pointed out in Theorem 4.1).

Henceforth, the ROM  $(A_N, B_N, C_N)$  will be assumed to be controllable and observable. It is not difficult to verify whether a given ROM satisfies this assumption via the usual finite dimensional space techniques; when the modal ROM is used, it is especially easy to verify, e.g., [12].

## 5.0 IMPLEMENTABLE FEEDBACK CONTROLLERS FOR DPS

The performance of the controlled DPS will usually be expressed in

terms that can be accomplished with ROM state feedback, e.g., pole placement or optimization of a performance index for the ROM. For example, when the modal ROM is used, the modes chosen would be the most critical (i.e., easily excited by whatever disturbance sources are present) modes; increased damping and/or stiffening of these modes would be the main goal of feedback control. We will suppose that the desired control law for the DPS (2.1) is given (or adequately approximated) by

$$f^*(t) = G_N v_N(t) \quad (5.1)$$

where the gain operator  $G_N$  can be selected via pole placement or optimal regulator methods, [26], applied to the ROM (4.10).

If this control law (5.1) could be implemented, the closed-loop system (2.1) and (5.1) would have the form (using (4.6)-(4.7)):

$$\dot{v}_N(t) = (A_N + B_N G_N) v_N(t) \quad (5.2)$$

$$\frac{\partial v_R(t)}{\partial t} = B_R G_N v_N(t) + A_R v_R(t) \quad (5.3)$$

where  $v(t) = v_N(t) + v_R(t)$ . The closed-loop stability of (5.2)-(5.3) would follow from the result:

#### THEOREM 5.1

Let the control law be given by (5.1). If  $(A_N, B_N)$  is controllable, then  $G_N$  can be chosen so that  $\Delta_N = \sigma_N - \epsilon > 0$  and

$$\|v(t)\| \leq M_0 M_N M^* e^{-\epsilon t} \|v_0\|, \quad t \geq 0 \quad (5.4)$$

where

$$M^* \equiv \begin{cases} \|P_R\| + \|P_N\| + \alpha_N \|P_R\| \|P_N\| & \text{if nonorthogonal projections } P_N \text{ and } P_R \text{ are used} \\ (1 + \alpha_N + \alpha_N^2)^{1/2} & \text{if orthogonal projections are used} \end{cases}$$

and  $\alpha_N \equiv \beta/\Delta_N$ ,  $\beta \equiv ||B_R G_N||$  (the control spillover coefficient),  
 $\bar{A}_N \equiv A_N + B_N G_N$  generates  $e^{\bar{A}_N t}$  with

$$||e^{\bar{A}_N t}|| \leq M_N e^{-\sigma_N t}, \quad t \geq 0 \quad (5.5)$$

and  $(M_0, \epsilon)$  are as in (2.2).

The proof of this theorem appears in Appendix III. Note that only control spillover  $\beta$  is present in (5.4); this is because direct access to  $v_N(t)$  has been assumed.

Of course, one rarely has direct access to the ROM states  $v_N(t)$ ; at best, one could try to recover them from the outputs of  $P = N$  sensors but observation spillover would interfere with this approach. Therefore, the control law (5.1) can not be exactly implemented (except in the special case that the observation spillover in (4.8) is zero). Instead, estimates of  $G_N v_N(t)$  must be generated from the sensor outputs  $y(t)$ . This can be done via the following implementable, finite dimensional controller:

$$\begin{cases} \dot{f}(t) = H_{11} y(t) + H_{12} z(t) & (5.6a) \\ \dot{z}(t) = F z(t) + H y(t) + E f(t), z(0) = 0 & (5.6b) \end{cases}$$

where  $\dim z = S \leq N = \dim v_N$ .

Let  $e_N(t) \equiv z(t) - T_N v_N(t)$  and, from (4.6), (4.8) and (5.6b), we obtain:

$$\begin{cases} \dot{e}_N(t) = F e_N(t) + H C_R v_R(t) \\ e_N(0) = -T_N P_N v_0 \end{cases} \quad (5.7)$$

when the following assumptions are made about  $T_N$  and  $E$ .

$$F T_N - T_N A_N + H C_N = 0 \quad (5.8)$$

$$E = T_N B_N \quad (5.9)$$

We will assume that the matrix F can have any desired stability margin  $\sigma_F$ , i.e.,

$$\|e^{Ft}\| \leq M_F e^{-\sigma_F t}, \quad t \geq 0 \quad (5.10)$$

This means that F can be chosen so that it shares no common eigenvalues with  $A_N$ ; therefore, by [27], Chapt. 8, there will exist a unique solution  $T_N$  of (5.8) and hence, E can be chosen to satisfy (5.9).

From (5.6a) and the definition of  $e_N(t)$ , we obtain:

$$f(t) = G_N v_N(t) + H_{12} e_N(t) + H_{11} C_R v_R(t) \quad (5.11)$$

when we make the assumption that  $H_{11}$  and  $H_{12}$  can be chosen to satisfy:

$$H_{11} C_N + H_{12} T_N = G_N \quad (5.12)$$

This last assumption, plus (5.10), can be satisfied when  $N - P \leq S \leq N$ , e.g., [28], but it can often be satisfied for  $S < N - P$ , e.g., [29]; the special case  $S = N$  is the usual full state estimator or Kalman filter based on the ROM, e.g., [30]. Therefore, the assumptions (5.10) and (5.12) can always be satisfied by some implementable controller (5.6) whose dimension S is less than N; the actual design of such a controller is a well-studied problem in finite dimensional space and will not concern us further; such a controller may be synthesized directly with a digital computer, [25], Chapt. 6. However, please note that the design of the controller (5.6), and all necessary assumptions, can be done entirely in terms of the ROM. This is a reasonable approach since the residual subsystem parameters ( $A_R, B_R, C_R$ ) may be much less accurately known than those for the ROM; in most cases only bounds on these parameters would be available in practice.

The question remains: How well can this implementable controller be made to perform? This is the subject of the next section.

## 6.0 CLOSED-LOOP OPERATION OF THE IMPLEMENTABLE ROM-BASED CONTROLLER

Even though the design of the controller (5.6) is based entirely on the ROM (4.10), this controller must operate in closed-loop with the actual DPS (4.6)-(4.8). Consequently, the residual subsystem (4.7) is involved in this operation via the control and observation spillover terms. It is a rare situation when these spillover terms are zero; this occurs if the actuator and sensor influence functions  $b_i$  and  $c_j$  lie entirely in the ROM subspace  $H_N$ . Most practical DPS problems will involve a certain amount of spillover even though it may be small; this can lead to degraded performance and, in some instances, instability [30].

Let  $q(t) \equiv [v_N^T(t) \ e_N^T(t)]^T$ . From (4.6)-(4.8), (5.7) and (5.11), we obtain the closed-loop system:

$$\begin{cases} \dot{q}(t) = \bar{A}_{11} q(t) + \bar{A}_{12} v_R(t) \\ \frac{\partial v_R(t)}{\partial t} = \bar{A}_{21} q(t) + \bar{A}_{22} v_R(t) \end{cases} \quad (6.1)$$

where

$$\bar{A}_{11} = \begin{bmatrix} A_N + B_N G_N & B_N H_{12} \\ 0 & F \end{bmatrix}$$

$$\bar{A}_{12} = \begin{bmatrix} B_N H_{11} & C_R \\ H & C_R \end{bmatrix}$$

$$\bar{A}_{21} = [B_R G_N \ B_R H_{12}] , \text{ and } \bar{A}_{22} = A_R + B_R H_{11} C_R$$

with the initial conditions  $q_0 = [p_N \ v_0, -p_N \ p_N \ v_0]^T$  and  $v_R(0) = p_R \ v_0$ . Define

$$\|q(t)\| = (\|v_N(t)\|^2 + \|e_N(t)\|^2)^{1/2}$$

We have the following useful result:

THEOREM 6.1

If no observation spillover is present ( $C_R = 0$ ) in (6.1), then  $F$  can be chosen so that  $\Delta_c = \sigma_F - \sigma_N > 0$  and

$$\|q(t)\| = (\|v_N(t)\|^2 + \|e_N(t)\|^2)^{1/2} \leq M_c e^{-\sigma_N t} \|q(0)\|, \quad t \geq 0 \quad (6.3)$$

where

$$M_c \leq M_N M_F \left(1 + \frac{\|B_N H_{12}\| \|T_N\|}{\Delta_c}\right) \quad (6.4)$$

and  $(M_N, \sigma_N)$ ,  $(M_F, \sigma_F)$  are as in (5.5) and (5.10), respectively.

The proof of this result appears in Appendix IV. Note that (6.4) is not the only bound for  $M_c$ ; in fact, the "sharper" we can make the bound on  $M_c$ , the better our later results become.

The next two results are the principal ones in this paper; they give sufficient conditions for exponential stability of the closed-loop system in terms of spillover bounds:

THEOREM 6.2

Assume that  $(A_N, B_N, C_N)$  is controllable and observable, that  $G_N$  is chosen in (5.12) so that  $\Delta_N = \sigma_N - \varepsilon > 0$ , and that  $F$  is chosen in (5.6b) so that  $\sigma_F > \sigma_N$ . If no observation spillover is present (i.e.,  $C_R = 0$ ), then, the closed-loop system (6.1) satisfies:

$$\|w(t)\| \leq M_c M_0 M^* e^{-\varepsilon t} \|w_0\|, \quad t \geq 0 \quad (6.5)$$

where

$$w(t) \equiv [v_N^T(t) \ e_N^T(t) \ v_R^T(t)]^T ,$$

$$v(t) = v_N(t) + v_R(t) ,$$

$$||w(t)|| \equiv (||v(t)||^2 + ||e_N(t)||^2)^{1/2} \text{ and}$$

$M^*$  is given by (5.4) in Theorem 5.1, and  $M_c$  is bounded as in (6.4) in Theorem 6.1.

The proof of this theorem appears in Appendix V.

### THEOREM 6.3

Let the open-loop system (2.1) be a strictly dissipative hyperbolic DPS (i.e.,  $\epsilon > 0$ ). Under the same assumptions as Theorem 6.2 except that observation spillover is present, the closed-loop system (6.1) stability margin is at least  $\epsilon'$  where

$$\epsilon' = \epsilon - M_c M_0 M^* \tau \quad (6.6)$$

when  $\tau \equiv ||B|| ||H_1|| C_R|| + ||H C_R||$  is sufficiently small  $\epsilon' > 0$ .

In that case,

$$||w(t)|| \leq M_c M_0 M^* e^{-\epsilon' t} ||w_0|| , \quad t \geq 0 \quad (6.7)$$

The proof of this theorem appears in Appendix VI. There is no point in considering such a result when (2.1) is not strictly dissipative hyperbolic, because  $\epsilon'$  could not ever be positive in (6.6).

### 7.0 APPLICATION: CONTROL OF FLEXIBLE STRUCTURES

Feedback control of vibrations in mechanically flexible structures has application to many current engineering problems. Control of flexible structures, in general, was addressed in [30], [36] and [38]. A new applica-

tion area - control of large aerospace structures - is very active at present; the current control theory trends in this area have been surveyed in [12].

The results obtained in this paper have direct application to control of flexible structures, in general, and large aerospace structures, in particular; this application will be discussed in this section in order to illustrate the use of the results of Sec. 6.0. This section will follow the format established in [38]; for clarity's sake, some of the format will be repeated here.

We consider the class of flexible systems that can be described by a generalized wave equation; this represents the idealization of many mechanically flexible structures. The generalized wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} + A_0 u = F \quad (6.1)$$

which relates the vector of displacements  $u(x,t)$  of a body  $\Omega$ , a bounded open set with smooth boundary  $\partial\Omega$  in  $n$ -dimensional Euclidean space  $R^n$ , to the applied control forces  $F(x,t)$ . The operator  $A_0$  is a time-invariant, symmetric differential operator with compact resolvent and lower semibounded spectrum. The domain  $D(A_0)$  of  $A_0$  is dense in the Hilbert space  $L^2(\Omega)$  with  $(\cdot, \cdot)$  denoting the usual inner product and  $\|\cdot\|$  denoting the associated norm. The natural damping in the structure is modeled by the term  $b \frac{\partial u}{\partial t}$  where  $b$  is a non-negative constant. The control forces

$$F(x,t) = Bf(t) = \sum_{i=1}^M b_i(x) f_i(t) \quad \text{ORIGINAL PAGE IS OF POOR QUALITY} \quad (6.2)$$

are provided by  $M$  actuators with influence functions  $b_i(x)$ . The displacements are measured by  $P$  averaging sensors

$$y(t) = C_0 u(x,t) \quad (6.3)$$

where  $y_j(t) = \int_{\Omega} c_j(x) u(x,t) dx$  with  $j = 1, 2, \dots, P$ . The actuator and sensor functions  $b_i(x)$ ,  $c_j(x)$  are in  $L^2(\Omega)$  and normalized to have unit integral.



When the support of  $b_i(x)$  is in a small neighborhood of a point  $x_i$ , we say it is a point actuator and, similarly, we define a point sensor. The point actuator and point sensor situation is of special interest here. This class of distributed parameter systems includes interior and boundary control of vibrating strings, membranes, thin beams and thin plates. Although, only displacement sensors are considered in (6.3), this is not a restriction - more general sensors may be modeled.

It is well known ([25], p. 277) that the spectrum of  $A$  contains only isolated eigenvalues  $\lambda_k$  with corresponding eigenfunctions  $\phi_k$  such that

$$\lambda_1 \leq \lambda_2 \leq \dots$$

and  $A\phi_k = \lambda_k \phi_k$ . We will assume that  $\lambda_1$  is positive. The resonant mode frequencies  $\omega_k$  of the structure are given by  $\omega_k = (\lambda_k)^{1/2}$  and the corresponding eigenfunctions  $\phi_k$  are the mode shapes. Thus  $A$  satisfies

$$(A_0 u, u) \geq a \|u\|^2, \quad a > 0 \quad (6.4)$$

and has a square root  $A_0^{1/2}$ . Every vector  $u \in L^2(\Omega)$  has a unique representation

$$u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x) \quad (6.5)$$

where  $u_k = \int_{\Omega} u \phi_k dx$  and we define the orthogonal projections  $p_0^N, p_0^R$  by

$$\begin{aligned} p_0^N u &= \sum_{k=1}^N u_k \phi_k \\ p_0^R u &= \sum_{k=N+1}^{\infty} u_k \phi_k \end{aligned} \quad (6.6)$$

Let  $V$  be the domain of  $A_0$  and  $W$  be the domain of  $A_0^{1/2}$ . A new operator  $A$  is defined in  $H$  by

$$\begin{aligned} \text{a) } D(A) &= V \times W \equiv H_1 \\ \text{b) } A \begin{bmatrix} u \\ w \end{bmatrix} &= \begin{bmatrix} w \\ -bw - A_0 u \end{bmatrix} \quad \text{for } u \in V, w \in W \end{aligned} \quad (6.7)$$

The energy inner product  $(\cdot, \cdot)_E$  is defined on  $H_1$  by

$$\left( \begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \right)_E \equiv (A_0^{1/2} u_1, A_0^{1/2} u_2) + (w_1, w_2) \quad (6.8)$$

for  $u_1, u_2 \in V$  and  $w_1, w_2 \in W$ , and the Hilbert space  $H$  is defined as the closure of  $H_1$  in this energy inner product. The associated energy norm is denoted by  $\|\cdot\|_E$  and is a measure of the total potential and kinetic energy in  $(u, \frac{\partial u}{\partial t})$  where  $u$  is a solution of (6.1). Let  $v = [u^T, \frac{\partial u}{\partial t}^T]^T$  be in  $H$  and write (6.1), (6.2) and (6.3) as

$$\begin{cases} \frac{\partial v}{\partial t} = Av + Bf, & v_0 \in H_1 \\ y = Cv \end{cases} \quad (6.9)$$

where  $B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}$  and  $C = [C_0 \ 0]$ ; this is in the form (2.1).

It is easy to see that

$$v = \sum_{k=1}^{\infty} \begin{bmatrix} u_k \\ \dot{u}_k \end{bmatrix} \phi_k \quad (6.10)$$

and  $\|v\|_E^2 = \sum_{k=1}^{\infty} [\lambda_k u_k^2 + \dot{u}_k^2]$ . Also,  $A$  generates a  $C_0$  semigroup  $U(t)$

by [11] p. II-59. The following result, due to Goldstein and Rosenkrans [39], gives conditions under which this semigroup  $U(t)$  satisfies (2.2) and hence (6.9) is a (strictly) dissipative hyperbolic system:

$$\text{if } b < 2a \text{ then} \quad (6.11)$$

$$\|v(t)\|_E^2 \leq \gamma e^{-bt} \|v(0)\|_E^2, \quad t \geq 0$$

where  $\gamma = [1+(b/2a)][1-(b/2a)]^{-1}$ . Consequently,  $U(t)$  satisfies (2.2) with  $M_0 = (\gamma)^{1/2}$  and  $\varepsilon = b/2$ . Note that the hypothesis for this result means that  $\varepsilon < a$ , i.e. the damping must be smaller than the square of the lowest

mode frequency. Another bound of the type (6.11) is available in [36] without the restriction  $\epsilon < a$ ; however, the damping coefficient  $b$  is not quite so directly related to  $\epsilon$  in (2.2). The bound used above will be very convenient to illustrate our results.

Recalling (6.6), we define the projections  $P_N, P_R$  on  $H$  by

$$\begin{aligned} P_N \begin{bmatrix} u \\ w \end{bmatrix} &\equiv \begin{bmatrix} p_o^N u \\ p_o^N w \end{bmatrix} \\ P_R \begin{bmatrix} u \\ w \end{bmatrix} &\equiv \begin{bmatrix} p_o^R u \\ p_o^R w \end{bmatrix} \end{aligned} \quad (6.12)$$

and note that they are orthogonal in the energy inner product and that  $H_N = P_N H$  and  $H_R = P_R H$  form reducing subspaces of  $H$ , as described in Sec. 4.0. In fact,  $H_N$  is a modal subspace, and  $v_N = P_N v$  and  $v_R = P_R v$  satisfy (4.1), (4.4)-(4.5). Consequently, the system (6.9) partitions into the form (4.6)-(4.8), and the corresponding modal ROM is given by (4.10) where  $A_N, B_N, C_N$  may be identified with the matrices

$$\begin{bmatrix} 0 & I_N \\ -\Lambda_N & -bI_N \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_N^0 \end{bmatrix}, \quad [C_N^0 \ 0], \quad \text{respectively, with } \Lambda_N = \text{diag} [\lambda_1, \dots, \lambda_N],$$

$\lambda_N], B_N^0 = [(b_j, \phi_k)]^T$ , and  $C_N^0 = [(c_j, \phi_k)]$ . Thus, the controllability and observability of the ROM can be determined from  $(\Lambda_N, B_N^0, C_N^0)$  when the damping  $b$  is small ([38], Theorem 2.1). Of course, it is quite easy to assess the controllability and observability of this latter system and the conditions depend on the interaction of the actuator-sensor influence functions with the mode shapes  $\phi_1, \dots, \phi_N$ .

If the ROM is controllable and observable, the usual finite dimensional, state space controller can be designed based on the modal ROM (e.g. [26]). It has the form:

$$f(t) = G_N \hat{v}_N(t) \quad (6.13)$$

$$\begin{cases} \dot{\hat{v}}_N(t) = A_N \hat{v}_N(t) + B_N f(t) + K_N(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C_N \hat{v}_N(t), \hat{v}_N(0) = 0 \end{cases} \quad (6.14)$$

This is a particular case of (5.6) with dimension  $S=2N$  (twice the number of modes in the ROM) and parameters:  $H_{11}=0$ ,  $H_{12}=G_N$ ,  $F = A_N - K_N C_N$ ,  $H=K_N$  and  $E=B_N$ . Also, note that  $e_N(t) = \hat{v}_N(t) - v_N(t)$ , i.e.  $T_N = I_{2N}$ . Since ROM is controllable and observable,  $G_N$  and  $K_N$  can be chosen to achieve (5.5) and (5.10) with any desired stability margins  $\sigma_N$  and  $\sigma_F$ .

Thus, (6.13)-(6.14) represents an implementable (finite dimensional) modal controller for the flexible structure (6.1)-(6.3), or equivalently (6.9). It is based on a modal ROM, i.e. it is designed to control the first  $N$  modes of the structure (of course, different sets of  $N$  modes could have been used - it is not necessary that they be the first  $N$  modes). The basic assumption made in synthesizing such a modal controller is that the  $N$  modes chosen for control will adequately represent the important vibrational behavior of the whole flexible structure; this assumption can often be satisfied in practice, and it is the basis for many structure controllers.

Having synthesized an implementable modal controller to adequately tailor the vibrational response of  $N$  critical modes in the flexible structure, we must not think we are done. The closed-loop behavior of our controller is affected by the residual modes in  $H_R$  for which we have not designed, as well as the controlled modes in  $H_N$  for which we have designed. These interactions occur through the control and observation spillover terms defined in Sec. 4.0 and can drastically alter the desired performance of the controlled structure; in fact, an example is presented in [30] where the closed-loop system becomes unstable.

Consequently, the analysis of closed-loop behavior discussed in Sec. 6.0 is especially pertinent. The principal results Theorems 6.2-6.3 give an indication of how much spillover can be tolerated in the closed-loop system. In the context of flexible structures, Theorem 6.2 may be compared with the results of [38] where observation spillover was assumed negligible (however, note that no damping was present in [38]). Combining the results of Theorems 6.2 and 6.3 for this application, we find that the closed-looped system, consisting of the flexible structure (6.1)-(6.3) and the finite dimensional modal controller (6.13)-(6.14), will remain exponentially stable with stability margin  $\epsilon'$  given by (6.6), if the observation spillover coefficient  $\Gamma$  is sufficiently small to make  $\epsilon' > 0$ , or equivalently:

$$\Gamma < \epsilon / M_C M_O M^* \quad (6.15)$$

In this application,  $\epsilon = \frac{b}{2}$  where  $b < 2a$  in (6.1) and (6.4),  $M_O$  is given by

$$M_O = \left( \frac{2a+b}{2a-b} \right)^{1/2} \quad (6.16)$$

Also, from (5.4) since orthogonal projections are used:

$$M^* = (1 + \alpha_N + \alpha_N^2)^{1/2} \quad (6.17)$$

where  $\alpha_N = \beta / \Delta_N$  and  $\Delta_N = \sigma_N - \epsilon$ , and, from (6.4),

$$M_C \leq M_N M_F \left( 1 + \frac{|B_N^c G_N|}{\Delta_C} \right) \quad (6.18)$$

where  $\Delta_C = \sigma_F - \sigma_N$  and  $M_N, M_F$  come from (5.5) and (5.10) respectively.

The "single bar" norm denotes the matrix norm compatible with the euclidean norm. Finally, the control and observation spillover coefficients  $\beta$  and  $\Gamma$  are given by

$$\beta = \|B_R G_N\|_E \leq \left( \sum_{i=1}^M \sum_{k=N+1}^{\infty} (b_i, c_k)^2 \right)^{1/2} \|G_N\| \quad (6.19)$$

and

$$\Gamma = \|K_N C_R\|_E \leq |K_N| \left( \sum_{j=1}^P \sum_{k=N+1}^{\infty} (c_j, \phi_k)^2 \right)^{1/2} \quad (6.20)$$

It may seem like quite a complicated business to obtain all the data to check (6.15); however, most of the information can be obtained from the finite dimensional ROM and its corresponding controller, and the rest comes from the structure properties  $a, b$ . The spillover coefficients (6.19)-(6.20) depend on the controller gains  $G_N, K_N$  and the actuator-sensor influence on the residual modes. Consequently, (6.15) gives an indication of how much spillover can be tolerated in the closed-loop system.

Additionally, (6.15) gives the control system designer an aid to redesign the system for better overall stability; quite simply, either the controller gains must be reduced or the actuator-sensor influence functions  $b_i, c_j$  must be chosen to minimize their influence on the residual modes. Of course, this latter could be accomplished if one had completely free choice of  $b_i, c_j$ ; one would choose them to be in  $H_N$  which is orthogonal to  $H_R$  and both  $\Gamma$  and  $\beta$  would be zero. Please note, for a flexible structure this would mean the influence functions were linear combinations of the controlled modes only. Yet, in practice, most actuators and sensors have a very localized influence (they are nearly point devices), and this practical constraint would make it very unlikely that zero spillover could be achieved. Nonetheless, adjustments in the location of actuators and sensors could be made in order to reduce the spillover. Further, one could reduce spillover by controlling more modes at the cost of increasing the controller complexity and corresponding computational burden on the on-line computer. Other approaches to spillover reduction and compensation (e.g. prefiltering) are surveyed in [12]. The apriori bound (6.15) can be used to assess the trade-offs in these approaches for flexible structure control.

## 8.0 CONCLUSIONS

In order to better understand the behavior of highly oscillatory distributed parameter phenomena such as mechanically flexible structures, the class of dissipative hyperbolic distributed parameter systems (DPS) has been introduced. The fundamental property of such DPS is that the open-loop system (2.1) must be exponentially stable, i.e., satisfy (2.2) for  $\epsilon \geq 0$ . Conditions have been presented in Section 3 for exponentially stable operation of the infinite dimensional controller, (3.2) and (3.4) or (3.6), in closed-loop with the DPS; however, the principal concern of this paper has been with finite dimensional controllers for the DPS. Such controllers are practical in the sense that they can be implemented with a small number of actuators and sensors and one, or more, small on-line computers.

The finite dimensional controllers (5.6) developed here in Section 5 are based on reduced-order models (ROMs), i.e., finite dimensional approximations, of the actual DPS. The class of ROMs used here has been derived in Section 4 from reducing subspaces; a large subclass of reducing subspaces contains the modal subspaces which have been used extensively in flexible structure modeling and control. The controller synthesis is done entirely in terms of the ROM.

Since the finite dimensional controllers developed in Section 5 must operate in closed-loop with the actual DPS and not just the ROM, it is crucial to analyze the spillover, i.e., controller interaction with the residual (unmodelled) part of the DPS. One cannot expect to be able to do this in utmost detail since the residuals of the DPS are the part that is least known; however, spillover bounds can be prescribed for stable operation of the closed-loop system and this has been done in Section 6. The principal

results are Theorems 6.2 - 6.3 ; the value of these results is that the stability conditions may be checked using only knowledge of the ROM and bounds,  $\beta$  and  $\Gamma$ , on the spillover present in the open-loop DPS. The bounds  $\beta$  and  $\Gamma$  may be calculated directly, or estimated where necessary, from the ROM projections of the actuator and sensor influence functions  $b_i$  and  $c_j$ . When the spillover bounds are small enough to satisfy the stability conditions, the finite dimensional controller will function to enhance the behavior of the ROM subsystem of the DPS without causing any instability in the residual subsystem. When the stability conditions are not satisfied, the system may be stable or it may not; however, this should be a warning that some spillover compensation be added and the conditions rechecked.

Several closely related DPS topics seem to require further investigation:

- (1) sharper spillover bounds
- (2) bounds obtained from singular perturbations
- (3) nonreducing subspaces and model error
- (4) nonlinear control
- (5) adaptive control

Since the spillover bounds have been obtained from regular perturbations, they may be sharpened by making tighter estimates, e.g., for  $M_c$  in (6.4); connections need to be made between our results and other stability results for large scale and DPS obtained by Siljak [31] and Michel and Miller [32]. In addition, bounds obtained for DPS via singular perturbations look promising for many applications, e.g., [33].

Many ROMs are projections onto nonreducing subspaces; for example, when approximate modes are substituted for exact modes, the approximate modal



subspaces are not reducing subspaces. In the case of nonreducing subspaces, the controller interaction with the residual subsystem can occur through non-zero model error terms  $A_{NR} = P_N A P_R$  and  $A_{RN} = P_R A P_N$ , as well as the spillover terms. The bounds obtained here for spillover should be extended to model error bounds, as well.

Finally, preliminary results in nonlinear and bilinear control of DPS have been obtained in [34]-[36] and the surface has barely been scratched in adaptive (or self-tuning) control of DPS, e.g., the survey [37]. Much more remains to be done in both of these areas.

# REFERENCES

1. John, F., Partial Differential Equations (3rd Edition), Springer-Verlag, NY, 1978.
2. Lions, J. L., Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, NY, 1971.
3. Lions, J. L., Some Aspects of the Optimal Control of DPS, SIAM, Philadelphia, PA, 1972.
4. Ray, W. H., and D. Lainiotis (eds.) DPS - Identification, Estimation and Control, Marcel Dekker, NY, 1978.
5. Aziz, A., et.al. (eds.), Control Theory of Systems Governed by Partial Differential Equations, Academic Press, NY, 1977.
6. Curtain, R., and A. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic Press, NY, 1977.
7. Ladas, G., and V. Lakshmikantham, Differential Equations in Abstract Spaces, Academic Press, NY, 1972.
8. Robinson, A., A Survey of Optimal Control of DPS, Automatica, 7(1971), pp. 371-388.
9. Russell, D., Controllability and Stability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions, SIAM Rev., 20(1978), pp. 371-388.
10. Balas, M., Disturbance Accommodating Control of DPS: An Introduction, Int'l. J. Modeling and Simulation, Jan. 1980 (to appear).
11. Gran, R., A Survey of the Large Structures Control Problem, Proc. of IEEE, CDC, Ft. Lauderdale, FL, 1979.
12. Balas, M., Some Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams, Proc. JACC, Denver, CO, 1979; to appear in revised form in IEEE Trans. Autom. Contr.
13. Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Univ. of Maryland, College Park, MD, Math Lecture Notes #10, 1974.
14. Goldstein, J., Semigroups of Operators and Applications, Tulane Univ., New Orleans, LA, Math Lecture Notes, 1979.
15. Balas, M., Feedback Control of Wave Propagation in Symmetric Hyperbolic PDE Systems, Proc. of 13th Asimolar Conf., Circuits, Systems and Computers, Pacific Grove, CA, 1979.

16. Friedman, A., Partial Differential Equations, Holt, Rinehart and Winston, NY, 1969.
17. Triggiani, R., On the Stabilizability Problem in Banach Space, J. Math. Anal. Appl., 52(1975); pp. 383-403.
18. Sakawa, Y., and T. Matsushita, Feedback Stabilization of a Class of Distributed Systems and Construction of a State Estimator, IEEE Trans. on Autom. Contr., AC-20(1975), pp. 748-753.
19. Balas, M., Feedback Control of Linear Diffusion Processes, Int. J. Contr., 29(1979), pp. 523-533.
20. Athans, M., Toward a Practical Theory for DPS, IEEE Trans. on Autom. Contr., AC-15(1970), pp. 245-247.
21. Lax, P., and R. Phillips, Scattering Theory, Academic Press, NY, 1967.
22. Gibson, S., A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Feedback, SIAM J. Contr. & Opt., (to appear).
23. Strang, G., and G. Fix, An Analysis of the Finite Element Method, Prentice-Hall, NJ, 1973.
24. Oden, J. T., and J. Reddy, An Introduction to the Mathematical Theory of Finite Elements, John Wiley and Sons, NY, 1976.
25. Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, NY, 1966.
26. Kwakernaak, H., and R. Sivan, Linear Optimal Control Systems, J. Wiley & Sons, NY, 1972.
27. Gantmacher, F., The Theory of Matrices, Vol. I, Chelsea, NY, 1959.
28. Luenberger, D., An Introduction to Observers, IEEE Trans. on Autom. Contr., AC-16(1971), pp. 596-602.
29. Roman, J., and T. Bullock, Design of Minimal Order Stable Observers for Linear Functions of the State Via Realization Theory, IEEE Trans. on Autom. Contr., AC-20(1975), pp. 613-622.
30. Balas, M., Feedback Control of Flexible Systems, IEEE Trans. on Autom. Contr., AC-23(1978), pp. 673-679.
31. Siljak, D. D., Large Scale Dynamic Systems: Stability and Structure, Elsevier-North Holland, Inc., NY, 1977.
32. Michel, A., and R. Miller, Qualitative Analysis of Large Scale Dynamical Systems, Academic Press, NY, 1977.

33. Balas, M., Singular Perturbations and Model Reduction Methods for Large Scale and Distributed Parameter Systems, Proc. of 17th Allerton Conf. on Comm., Contr., and Computing, Monticello, IL, 1979.
34. Ball, J., and M. Slemrod, Feedback Stabilization of Distributed Semilinear Control Systems, Appl. Math. and Opt., 5(1979), pp. 169-179.
35. Ball, J., and M. Slemrod, Nonharmonic Fourier Series and the Stabilization of Distributed Semilinear Control Systems, Comm. Pure and Appl. Math., 32(1979), pp. 555-587.
36. Balas, M., Control of Flexible Structures in the Presence of Certain Nonlinear Disturbances, Proc. of Joint Autom. Contr. Conf., Denver, CO, 1979.
37. Balas, M., and C. R. Johnson, Jr., Adaptive Control of DPS: The Ultimate Reduced-Order Problem, Proc. of IEEE Contr. & Decision Conf., Ft. Lauderdale, FL, 1979.
38. Balas, M., Modal Control of Certain Flexible Dynamic Systems, SIAM J. Contr. and Opt., 16(1978), pp. 450-462.
39. J. Goldstein and S. Rosencrans, Energy Decay and Partition for Dissipative Wave Equations, J. Differential Equations, to appear 1980.

# APPENDIX I: PROOF OF THEOREM 3.1

Since  $(A, B)$  and  $(A^*, C^*)$  are exponentially stabilizable, there exist bounded  $G$  and  $H$  such that  $A + B G$  and  $A^* + C^* H$  generate  $U_1(t)$  and  $V(t)$ , respectively;  $U_1(t)$  satisfies (3.5) and  $V(t)$  satisfies

$$\|V(t)\| \leq M_2 e^{-\epsilon_2 t}, \quad t \geq 0 \quad (I.1)$$

where  $M_2 \geq 1$  and  $\epsilon_2 > 0$ . Choose  $K = -H^*$  and from [21], p. 251,  $A - K C$  generates  $U_2(t)$  where  $U_2(t) = V^*(t)$ ; consequently,  $\|U_2(t)\| = \|V^*(t)\| = \|V(t)\|$ . Therefore,

$$\|e(t)\| \leq M_2 e^{-\epsilon_2 t} \|v_0\| \quad (I.2)$$

Also, from (3.3)-(3.4) and (2.1),

$$\frac{\partial v(t)}{\partial t} = (A + BG) v(t) + BG e(t) \quad (I.3)$$

or, from (3.5) and (I.2),

$$\|v(t)\| \leq M_1 e^{-\epsilon_1 t} \left(1 + M_2 \int_0^t e^{\Delta \tau} d\tau \|BG\|\right) \|v_0\| \quad (I.4)$$

where  $\Delta = \epsilon_1 - \epsilon_2$ . Let  $M_E \equiv M_1 M_2$  and  $\delta \equiv \min(\epsilon_1, \epsilon_2)$ .

If  $\Delta > 0$ ,  $\|v(t)\| \leq M_E e^{-\delta t} \|v_0\| \left(1 + \|BG\| \frac{1 - e^{-\Delta t}}{\Delta}\right)$ . If  $\Delta < 0$ ,

$$\|v(t)\| \leq M_E e^{-\delta t} \|v_0\| \left(1 + \|BG\| \frac{1 - e^{-\Delta t}}{\Delta}\right).$$

In either case, for  $t \geq 0$ ,

$$\|v(t)\| \leq M_E e^{-\delta t} \|v_0\| (1 + \|BG\|/|\Delta|) \quad (I.5)$$

From (I.2), for  $t \geq 0$ ,

$$\|e(t)\| \leq M_E e^{-\delta t} \|v_0\| \quad (I.6)$$

Therefore, the closed-loop system is exponentially stable.

## APPENDIX II: PROOF OF THEOREM 4.1

By the definition of reducing subspaces (4.2c and d),  $A_{NR} \equiv P_N A P_R = 0$  and  $A_{RN} \equiv P_R A P_N = 0$  and (2.1) becomes (4.6)-(4.8). Since the definition of reducing subspaces is equivalent to  $P_N$  and  $P_R$  commuting with  $A$  (see 4.3b), then, by [25], p. 173, they also commute with the resolvent operator  $R(\lambda, A)$  of  $A$  for any  $\lambda$  in the resolvent set; because  $A$  generates a semigroup, the resolvent is non-empty. But  $R(\lambda, A)$  is the (unique) Laplace transform of the transform of the semigroup  $U(t)$ , e.g. [25], p. 482; therefore, the projections must commute with  $U(t)$ .

Let  $U_R(t) \equiv P_R U(t) P_R$  then  $U_R(t)$  is a  $C_0$  semigroup on  $H_R$  because  $U_R(t) U_R(t') = P_R U(t) P_R U(t') P_R = P_R U(t) U(t') P_R = P_R U(t + t') P_R = U_R(t + t')$  for  $t, t' \geq 0$ ; this uses the fact that  $P_R$  commutes with  $U(t)$ . Of course,  $U_R(0) = P_R$  which is the identity on  $H_R$  and the strong continuity follows from that of  $U(t)$ . It is also clear that  $A_R$  generates  $U_R(t)$  because, for  $v$  in  $H_R$ ,  $\lim_{t \rightarrow 0} \frac{U_R(t)v - v}{t} = \lim_{t \rightarrow 0} P_R \frac{U(t)v - v}{t} = P_R \lim_{t \rightarrow 0} \frac{U(t)v - v}{t} = P_R Av = P_R A P_R v = A_R v$  by the continuity of  $P_R$ . The growth property (4.3) follows from (2.2) and  $\|U_R(t)\| = \|P_R U(t)\| \leq \|P_R\| \|U(t)\|$  because  $P_R$  commutes with  $U(t)$ .

### APPENDIX III: PROOF OF THEOREM 5.1

Since  $(A_N, B_N)$  is controllable and finite dimensional,  $G_N$  can be chosen so that  $\sigma_N > \epsilon$ . From (5.2)-(5.3) and (5.5),

$$||v_N(t)|| \leq M_N e^{-\sigma_N t} ||v_N(0)|| \quad (\text{III.1})$$

$$||v_R(t)|| \leq M_R e^{-\epsilon t} (||v_0|| + \int_0^t e^{\epsilon \tau} ||B_R G_N|| ||v_N(\tau)|| d\tau) \quad (\text{III.2})$$

because  $U_R(t) v_R(0) = P_R U(t) v_0$  from Theorem 4.1. Combining III.1 and III.2, obtain

$$||v_R(t)|| \leq M_R e^{-\epsilon t} (1 + M_N (1 - e^{-\Delta_N t}) \alpha_N ||P_N||) ||v_0|| \quad (\text{III.3})$$

Therefore, using (III.1) and (III.3),

$$||v(t)|| \leq ||v_N(t)|| + ||v_R(t)|| \leq e^{-\epsilon t} [M_R + M_N (e^{-\Delta_N t} + M_R (1 - e^{-\Delta_N t}) \alpha_N) ||P_N||] ||v_0|| \quad (\text{III.4})$$

Note that (III.4) is the same as Triggiani's bound in [17], Theorem 6.1.

From (III.4), obtain (5.4) for nonorthogonal projections because  $e^{-\Delta_N t} + M_R \alpha_N (1 - e^{-\Delta_N t}) \leq e^{-\Delta_N t} + M_R \alpha_N \leq 1 + M_R \alpha_N$  and  $M_R = M_0 ||P_R|| \geq 1$  from

Theorem 4.1. However, when the projections are orthogonal, then  $||v_R(t)|| \leq M_R e^{-\epsilon t} (||v_R(0)|| + M_N \alpha_N ||v_N(0)||)$  and  $||v(t)||^2 = ||v_N(t)||^2 + ||v_R(t)||^2$

$$\leq (M_0 M_N e^{-\epsilon t})^2 [(||v_R(0)|| + \alpha_N ||v_N(0)||)^2 + ||v_N(0)||^2]$$

$$\leq (1 + \alpha_N + \alpha_N^2) (M_0 M_N e^{-\epsilon t})^2 [||v_N(0)||^2 + ||v_R(0)||^2]$$

$$||v(t)|| \leq (1 + \alpha_N + \alpha_N^2)^{1/2} M_0 M_N e^{-\epsilon t} ||v_0||$$

because  $(a + \alpha b)^2 + a^2 \leq (1 + \alpha + \alpha^2) (a^2 + b^2)$  which is Lemma 3.1 in [19].

APPENDIX IV: PROOF OF THEOREM 6.1

With  $C_R = 0$  in (6.1),

$$q(t) = e^{\bar{A}_{11}t} q_0 \quad (\text{IV.1})$$

From (5.5) and (5.10), it is easy to obtain

$$||e_N(t)|| \leq M_F e^{-\sigma_F t} ||T_N v_N(0)|| \quad (\text{IV.2})$$

and

$$||v_N(t)|| \leq M_N e^{-\sigma_N t} [||v_N(0)|| + \int_0^t e^{\Delta_N \tau} ||B_N H_{12}|| ||e_N(\tau)|| d\tau] \quad (\text{IV.3})$$

But, using (IV.2), (IV.3) becomes

$$||v_N(t)|| \leq M_N M_F e^{-\sigma_N t} \left[ 1 + \frac{||B_N H_{12}|| ||T_N||}{\Delta_c} \right] ||v_N(0)|| \quad (\text{IV.4})$$

because  $F$  can be chosen so that  $\Delta_c > 0$ . From (IV.2) and (IV.4), the desired result follows because  $M_R$  and  $M_N$  are not less than unity and  $\Delta_c > 0$ .



APPENDIX V: PROOF OF THEOREM 6.2

Let  $C_R = 0$  in (6.1) and note that  $\bar{A}_{12} = 0$  and  $\bar{A}_{22} = A_R$ . From (6.1), obtain

$$\begin{cases} \dot{q}(t) = A_{11} q(t) \\ \frac{\partial v_R(t)}{\partial t} = \bar{A}_{21} q(t) + A_R v_R(t) \end{cases} \quad (V.1)$$

where  $\|q(t)\|$  satisfies (6.3) in Theorem 6.1 by appropriate choices of  $F$  and  $G_N$ . From Theorem 4.1,  $A_R$  generates the semigroup  $U_R(t)$  with growth property (4.9). Using this in (V.1) produces

$$\|v_R(t)\| \leq M_R e^{-\epsilon t} (\|v_0\| + \int_0^t e^{\epsilon \tau} \|\bar{A}_{21}\| \|q(\tau)\| d\tau) \quad (V.2)$$

where  $\|\bar{A}_{21}\| = \|B_R G_N\|$

Since (6.3) is similar to (III.1) and (V.2) is similar to (III.2), the proof of this theorem - Theorem 6.2 - can be carried out in the same manner as the proof of Theorem 5.1 in Appendix III (where  $q(t)$  takes the place of  $v_N(t)$  and  $M_0$  replaces  $M_N$ .)

# APPENDIX VI: PROOF OF THEOREM 6.3

Let  $w(t) = [q^T(t) \ v_R^T(t)]^T$  and, from (6.1), obtain

$$w(t) = \bar{A}_0 w(t) + \Delta \bar{A}_0 w(t) \quad (VI.1)$$

where  $\bar{A}_0 = \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & A_R \end{bmatrix}$  and  $\Delta \bar{A}_0 = \begin{bmatrix} 0 & \bar{A}_{12} \\ 0 & B_R^H H_{11} C_R \end{bmatrix}$

From Theorem 6.2,  $\bar{A}_0$  generates a semigroup  $S_0(t)$  with growth property, from (6.5):

$$\|S_0(t)\| \leq M_E e^{-\epsilon t} \quad (VI.2)$$

where  $M_E \equiv M_C M_0 M^*$ . Also, from (6.1),

$$\|\Delta \bar{A}_0\| = (\|\bar{A}_{12}\|^2 + \|B_R^H H_{11} C_R\|^2)^{1/2} \leq \Gamma \quad (VI.3)$$

Since  $\Delta \bar{A}_0$  is bounded,  $\bar{A}_0 + \Delta \bar{A}_0$  generates a semigroup  $S(t)$  (e.g., [13] p. 80 or [6] p. 210); therefore

$$w(t) = S(t) w(0) \quad (VI.4)$$

However, from (VI.1),

$$w(t) = S_0(t) w(0) + \int_0^t S_0(t-\tau) \Delta \bar{A}_0 w(\tau) d\tau \quad (VI.5)$$

Taking bounds in (VI.5) and using (VI.3), obtain

$$\|w(t)\| \leq M_E e^{-\epsilon t} [\|w(0)\| + \Gamma \int_0^t e^{\epsilon \tau} \|w(\tau)\| d\tau] \quad (VI.6)$$

Let  $h(t) \equiv e^{\epsilon t} \|w(t)\|$  and, from (VI.5), obtain

$$h(t) \leq M_E h(0) + \Gamma \int_0^t h(\tau) d\tau \quad (VI.7)$$

From the Gronwall inequality (e.g., [6] p. 124 or [14] p. III-45) applied to (VI.7),

$$h(t) \leq M_E e^{\epsilon t} h(0) \quad (\text{VI.8})$$

or, equivalently,

$$||w(t)|| \leq e^{-\epsilon' t} ||w(0)|| \quad (\text{VI.9})$$

where  $\epsilon'$  is given by (6.6) and this is the desired result (6.7). Clearly, the semigroup  $S(t)$  is exponentially stable when  $\epsilon'$  is positive.

**B.** FINITE ELEMENT MODELS AND FEEDBACK CONTROL  
OF FLEXIBLE AEROSPACE STRUCTURES

by

Mark J. Balas

Electrical and Systems Engineering Department  
Rensselaer Polytechnic Institute  
Troy, NY 12181

(518) 270-6485

# ABSTRACT

Large flexible aerospace structures, such as the solar power satellite, are distributed parameter systems with very complex continuum descriptions. This paper investigates the use of finite element methods to produce reduced-order models and finite dimensional feedback controllers for these structures. Our main results (Section 5.0) give conditions under which stable control of the finite element model will produce stable control of the actual structure.

## 1.0 Introduction

Large flexible aerospace structures, such as the solar power satellite<sup>[1]</sup>, are distributed parameter systems with very complex continuum descriptions. Often, their dynamics are modeled by large-scale systems of ordinary differential equations which are obtained via the finite element method for structures. To obtain implementable, i.e. finite dimensional, feedback controllers for these structures, we shall use this model as the basis for the control synthesis.

The question of closed-loop operation of such controllers within the actual structure is certainly a fundamental one. In this paper, we shall point out some connections between the finite element models and the closed-loop stability. These results will be used to give an answer to the following version of the above fundamental question: When does stable control of the finite element model ensure stable control of the actual structure?

For distributed parameter systems, in general, the finite element method (FEM) for time-dependent problems is very well explained in [2] Chapter 7, [3] Chapter 9, and the excellent survey [4]. The related approaches called "spectral methods" are described in [5]. The use of FEM for analysis of distributed parameter control problems involving interior and boundary controls has been considered, for example, in [6] - [10]; all of these references are concerned with parabolic (i.e. diffusion-like) distributed systems and, with the exception of [10], they are concerned with open-loop (i.e. non-feedback) control.

In contrast, our concern is feedback control of hyperbolic (i.e. highly oscillatory) distributed parameter systems and the use of FEM

to develop low-order finite dimensional controllers for such systems. Control approaches for structures have been surveyed in [11]-[13]. The most popular approach for structural control is the resolution of the structural vibration into normal modes, (semi) discretization of the distributed system by truncation of this modal expansion, and controller design based on the truncated modal model; this is called modal control. Modal analysis of structures has been described in many references but an enjoyable survey is [14]. The use of modal analysis in feedback controller design depends on a complete knowledge of the modal data which is rarely available for something as complex as a large aerospace structure. Instead, this modal data is approximated by finite element structural analysis computer codes (e.g. NASTRAN), and it is this approximate modal model that is used for the controller design, e.g. [15].

The stability of closed-loop operation of the finite dimensional modal controller in the infinite dimensional distributed parameter structure is questionable even when the actual modes are known, as shown by a simple example in [16]. The mechanism for instability is the combination of control and observation spillover in the residual (unmodeled) modes of the structure [13]. However, closed-loop stability bounds have been obtained for highly oscillatory distributed parameter systems with low-order finite dimensional controllers [17]. These bounds indicate the tolerable levels of spillover for stable operation, but they are based on "reducing" subspaces. For large aerospace structures, this corresponds to exact knowledge of the modal data used in the controller, although the residual modal data need not be so precisely known. This paper addresses the closed-loop stability question

for structure controllers based on finite element approximation of the modal data. The distributed parameter control problem for flexible structures is described in Section 2 and the finite element approximation in Section 3. The corresponding finite element modal controllers are presented in Section 4, and our main results on the closed-loop stability are Theorems 1 and 2 in Section 5.

## 2.0 The Distributed Parameter Control Problem for Flexible Structures

We consider the class of flexible structures that can be described by a generalized wave equation; this represents an idealization of many mechanically flexible structures. The generalized wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} + A_0 u = F \quad (2.1)$$

which relates the (vector of) displacements  $u(x,t)$  of a body  $\Omega$ , a bounded open set with smooth boundary  $\partial\Omega$  in  $n$ -dimensional Euclidean space  $R^n$ , to the applied control forces  $F(x,t)$ . The operator  $A_0$  is a time-invariant, self-adjoint differential operator with compact resolvent and lower semi-bounded spectrum. The domain  $D(A_0)$  of  $A_0$  is dense in the Hilbert space  $L^2(\Omega)$  with  $(\cdot, \cdot)_0$  denoting the usual inner product and  $\|\cdot\|_0$  denoting the associated norm. The natural damping in the structure is modeled by the term  $b \frac{\partial u}{\partial t}$  where  $b$  could be any bounded self-adjoint operator commuting with  $A_0$  but is taken here to be a non-negative constant, for convenience. The control forces

$$F(x,t) = B_0 f(t) = \sum_{i=1}^M b_i(x) f_i(t) \quad (2.2)$$

are provided by  $M$  actuators with influence functions  $b_i(x)$ . The displacements are measured by  $P$  averaging sensors



$$y(t) = c_0 u(x, t) \quad (2.3)$$

where  $y_j(t) = \int_{\Omega} c_j(x) u(x, t) dx$  with  $j = 1, 2, \dots, P$ . The actuator and sensor functions  $b_j(x)$ ,  $c_j(x)$  are in  $L^2(\Omega)$  and normalized to have unit integral. When the support of  $b_j(x)$  is in a small neighborhood of a point  $x_j$ , we say it is a point actuator and, similarly, we define a point sensor. The point actuator and point sensor situation is our principal interest here. This class of distributed parameter systems includes interior and boundary control of vibrating strings, membranes, thin beams and thin plates. Although, only displacement sensors are considered in (2.3), this is not a restriction - more general sensors may be modeled.

It is well known ([18], p. 277) that the spectrum of  $A_0$  contains only isolated eigenvalues  $\lambda_k$  with corresponding eigenfunctions  $\phi_k$  such that

$$\lambda_1 \leq \lambda_2 \leq \dots$$

and  $A_0 \phi_k = \lambda_k \phi_k$ . We will assume that  $\lambda_1$  is positive. The resonant mode frequencies  $\omega_k$  of the structure are given by  $\omega_k = (\lambda_k)^{1/2}$  and the corresponding eigenfunctions  $\phi_k$  are the mode shapes. Thus  $A_0$  satisfies

$$(A_0 u, u)_0 \geq a \|u\|_0^2, \quad a > 0 \quad (2.4)$$

and has a square root  $A_0^{1/2}$ . Every vector  $u \in L^2(\Omega)$  has a unique representation

$$u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x) \quad (2.5)$$

where  $u_k = (u, \phi_k)_0$  and we define the orthogonal projections  $P_0^N$ ,  $P_0^R$  by

$$P_0^N u = \sum_{k=1}^N u_k \phi_k$$

$$P_0^R u = \sum_{k=N+1}^{\infty} u_k \phi_k \quad (2.6)$$

Let  $V$  be the domain of  $A_0$  and  $W$  be the domain of  $A_0^{1/2}$ . A new operator  $A$  is defined in  $H$  by

$$\begin{aligned} \text{a) } D(A) &= V \times W \equiv H_1 \\ \text{b) } A \begin{bmatrix} u \\ w \end{bmatrix} &= \begin{bmatrix} -bw \\ -A_0 u \end{bmatrix} \quad \text{for } u \in V, w \in W \end{aligned} \quad (2.7)$$

The energy inner product  $(\cdot, \cdot)$  is defined on  $H_1$  by

$$\left( \begin{bmatrix} u_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ w_2 \end{bmatrix} \right) \equiv (A_0^{1/2} u_1, A_0^{1/2} u_2)_0 + (w_1, w_2)_0 \quad (2.8)$$

for  $u_1, u_2 \in V$  and  $w_1, w_2 \in W$ , and the Hilbert space  $H$ , called the state space, is defined as the closure of  $H_1$  in this energy inner product. The associated energy norm is denoted by  $||\cdot||$  and is a measure of the total potential and kinetic energy in  $(u, \frac{\partial u}{\partial t})$  where  $u$  is a solution of (2.1). Let  $v = [u^T, \frac{\partial u}{\partial t}^T]^T$  be in  $H$  and write (2.1), (2.2) and (2.3) as

$$\begin{cases} \frac{\partial v}{\partial t} = Av + Bf, & v_0 \in H_1 \\ y = Cv \end{cases} \quad (2.9)$$

where  $B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}$  and  $C = [C_0 \ 0]$ .

It is easy to see that

$$v = \sum_{k=1}^{\infty} \begin{bmatrix} u_k \\ \dot{u}_k \end{bmatrix} \phi_k \quad (2.10)$$

and  $||v||^2 = \sum_{k=1}^{\infty} [\lambda_k u_k^2 + \dot{u}_k^2]$ . Also,  $A$  generates a  $C_0$  semigroup  $U(t)$

by [19], p. II-59. The following result, due to Goldstein and Rosencrans [20], gives conditions under which this semigroup  $U(t)$  is (slightly) exponentially stable and hence (2.9) is a (strictly) dissipative hyperbolic system:

if  $b < 2a$  then

$$||v(t)|| \leq M_0 e^{-\epsilon t} ||v(0)||, \quad t \geq 0 \quad (2.11)$$

where  $M_0 = (\gamma)^{1/2}$ ,  $\epsilon = b/2$ , and  $\gamma = [1+(b/2a)][1-(b/2a)]^{-1}$ . Note that the hypothesis for this result means that  $\epsilon < a$ , i.e. the damping must be smaller than the square of the lowest mode frequency. Another bound of the type (2.11) is available in [21] without the restriction  $\epsilon < a$ ; however, the damping coefficient  $b$  is not quite so directly related to  $\epsilon$  in (2.11). The bound used above will be very convenient to illustrate our results.

Recalling (2.6), we define the projections  $P_N, P_R$  on  $H$  by

$$P_N \begin{bmatrix} u \\ w \end{bmatrix} \equiv \begin{bmatrix} P_O N_U \\ P_O N_W \end{bmatrix} \quad (2.12)$$

$$P_R \begin{bmatrix} u \\ w \end{bmatrix} \equiv \begin{bmatrix} P_O R_U \\ P_O R_W \end{bmatrix}$$

and note that they are orthogonal in the energy inner product and that  $H_N = P_N H$  and  $H_R = P_R H$  form reducing subspaces of  $H$ , as described in [17].

In fact,  $H_N$  is a modal subspace, and  $v_N = P_N v$  and  $v_R = P_R v$  satisfy:

$$\begin{cases} \frac{\partial v_N}{\partial t} = A_N v_N + B_N f \\ \frac{\partial v_R}{\partial t} = A_R v_R + B_R f \\ y = C_N v_N + C_R v_R \end{cases} \quad (2.13)$$

where  $A_N = P_N A P_N$ ,  $B_N = P_N B$ ,  $C_N = C P_N$ , and, correspondingly, for  $(A_R, B_R, C_R)$ . Note that  $A_N, B_N, C_N$  may be identified with the matrices  $\begin{bmatrix} 0 & I_N \\ -\Lambda_N & -bI_N \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ B_N^0 \end{bmatrix}$ ,  $[C_N^0 \ 0]$ , respectively, with  $\Lambda_N = \text{diag} [\lambda_1, \dots, \lambda_N]$ ,  $B_N^0 = [(b_j, \phi_k)]^T$ , and  $C_N^0 = [(c_j, \phi_k)]$ , and, if point sensors and actuators are used, then  $B_N^0$  and  $C_N^0$  involve only mode shapes  $\phi_k$  evaluated at the actuator and sensor locations. The terms  $B_R f$  and  $C_R v_R$  are called, respectively, control and observation spillover, and it is through these terms that the residual modes  $v_R$  are affected by any feedback controller which accepts the sensor outputs  $y$  and generates control commands  $f$ .

Modal controllers can be based on the reduced-order model (ROM) whose states are  $v_N$  and whose parameters are  $(A_N, B_N, C_N)$ ; such a model is obtained by ignoring (for the purpose of controller synthesis) the residual modes, i.e. setting  $B_R = 0$  and  $C_R = 0$ . The controllability and observability of the ROM can be determined from  $(A_N, B_N^0, C_N^0)$  when the damping  $b$  is small ([22], Theorem 2.1). Of course, it is quite easy to assess the controllability and observability of this latter system, and the conditions depend on the interaction of the actuator-sensor influence functions with the mode shapes  $\phi_1, \dots, \phi_N$ .

If the ROM is controllable and observable, the usual finite dimensional, state space controller can be designed based on the modal ROM (e.g. [23] or [24]). It has the form:

$$f(t) = G_N \hat{v}_N(t) \quad (2.13)$$

$$\begin{cases} \dot{\hat{v}}_N(t) = A_N \hat{v}_N(t) + B_N f(t) + K_N(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C_N \hat{v}_N(t), \hat{v}_N(0) = 0 \end{cases} \quad (2.14)$$

This controller has dimension  $2N$  (twice the number of modes in the ROM). Since the ROM is controllable and observable,  $G_N$  and  $K_N$  can be chosen to achieve any desired stability margins  $\sigma_N$  and  $\sigma_F$  for  $A_N + B_N G_N$  and  $A_N - K_N C_N$  respectively; optimal control, as well as pole placement techniques, may be used successfully for the controller design.

Thus, (2.13)-(2.14) represents an implementable (finite dimensional) modal controller for the flexible structure (2.1)-(2.3), or equivalently (2.9). It is based on a modal ROM, i.e. it is designed to control the first  $N$  modes of the structure (of course, different sets of  $N$  modes could have been used - it is not necessary that they be the first  $N$  modes). The basic assumption made in synthesizing such a modal controller is that the  $N$  modes chosen for control will adequately represent the important vibrational behavior of the whole flexible structure; this assumption can often be satisfied in practice, and it is the basis for most structure controllers.

Having synthesized an implementable modal controller to adequately tailor the vibrational response of  $N$  critical modes in the flexible structure, we must not think we are done. The closed-loop behavior of our controller is affected by the residual modes in  $H_R$  for which we have not designed, as well as the controlled modes in  $H_N$  for which we have designed. These interactions occur through the control and observation spillover terms defined in (2.13) and can drastically alter the desired performance of the controlled structure; in fact, an example is presented in [16] where the closed-loop system becomes unstable. Consequently, the closed-loop stability analysis presented in [17] is extremely important; these results give an indication of the amount of spillover that can be tolerated in the closed-loop system without loss of stability.

### 3.0 The Finite Element Approximation and Open-Loop Convergence

Unfortunately, all of the controller synthesis and closed-loop analysis presented so far is based on reducing subspaces  $H_N$  and  $H_R$  and, in particular, on exact knowledge of the system modes  $\phi_1, \dots, \phi_N$  used in the ROM; this modal data must be approximated in most cases due to the complexity of the actual partial differential equations and associated boundary conditions which describe the structure (2.1). Such approximations do not lead to reducing subspaces, in general; therefore, further discussion is necessary.

The most natural approach (and the one we shall use) seems to be to approximate the modal data required for the modal ROM ( $A_N, B_N, C_N$ ) via the finite element method (FEM) for time-dependent distributed parameter systems (2.9); the data required will be the ROM mode frequencies  $\omega_k$  and the corresponding mode shapes  $\phi_k(x)$  evaluated in neighborhoods of the actuator and sensor locations when point devices are used. Once this FEM approximated modal data is available, the controllability and observability of the ROM may be tested, and the FEM based modal controller may be synthesized using the approximate modal data in (2.13)-(2.14) as though it were exact. Although the FEM approach requires a great deal of "number-crunching" to obtain the approximate modal data, this data is needed for the open-loop structural analysis and is calculated anyway, regardless of control considerations. Also, the large scale computer codes necessary for such FEM calculations are available and in general use (e.g. NASTRAN and others).

This makes the FEM approach to structure controller synthesis appealing; however, the closed-loop stability analysis is much more diffi-

cult. The fundamental question to be answered is - when does stable control of the FEM model produce stable control of the actual structure? To begin to answer this question, we must give a more precise description of the FEM approach.

A precise description of the FEM approximation of time-dependent distributed parameter systems like (2.9) is given, for example, in [2] Chapter 7, [3] Chapter 9, or the survey article [4]. We point out that, historically, the FEM was developed specifically to numerically approximate the behavior of a given distributed parameter system or set of partial differential equations; in control terms, it was intended to give a close approximation of the dynamic behavior of open-loop systems like (2.9). However, we will use it in a very different way, namely to develop reduced-order models of the system (2.9) for synthesis of finite dimensional, feedback controllers to be used in closed-loop with the system (2.9). In the usual applications, the stability of the ROM plays a big role, and hyperbolic systems like (2.9) are notorious for their lack, or very low margin, of stability when the FEM is used. In our application, the stability of the ROM is less important because the feedback controller will be designed to improve its stability; more important to our application is the stability of the residual (unmodeled) part of the system over which we have no direct control.

We describe the FEM approach to finite-dimensional controller synthesis for the distributed parameter system (2.9). Let  $H_N$  be an increasing sequence of finite dimensional subspaces of the state space  $H$  for (2.9). Each subspace  $H_N$  has dimension  $N$ . To make life easier, we assume that each  $H_N$  is a subspace of  $D(A)$  so that its elements satisfy

the boundary conditions for  $A$ ; however, so called non-conforming elements may be used in the more general case. In the FEM, each subspace  $H_N$  consists of splines (i.e. piecewise-polynomial functions) of fixed degree defined over a mesh (usually, of triangles) laid out to approximately cover the structure  $\Omega$  (see [3] Chapter 6). No matter how irregular the shape of the boundary of  $\Omega$  such meshes can be fitted very closely; this is one of the principal assets of the FEM. To each mesh, a normalized mesh parameter  $h$  (where  $0 < h \leq 1$ ) is assigned so that the mesh is refined as  $h \rightarrow 0$  and the dimension  $N$  of the subspaces increases indefinitely.

### 3.1 The Galerkin Approximation

Let  $P_N$  be the orthogonal projection from  $H$  into  $H_N$ ; this is called the Galerkin projection. The corresponding orthogonal projection onto  $H_N^\perp$  is called  $P_R$  (i.e.  $P_R = I - P_N$ ). The "rate of convergence" of  $H_N$  to  $H$  is said to be of order  $q$  when

$$\|P_R v\| \leq K h^q \quad (3.1)$$

for  $v$  in  $H$  (or  $D(A)$ ); this rate is related to the ability of splines in  $H_N$  to interpolate functions in  $H$ . We shall not be concerned with the rate of convergence ; consequently we write (3.1) as

$$\lim_{N \rightarrow \infty} \|P_R v\| = 0 \quad \text{for } v \text{ in } H \quad (3.2)$$

and suppress the dependence on  $h$  for our discussion.

Let  $\psi_1(x), \dots, \psi_N(x)$  form a basis in  $H_N$  (i.e. they are linearly independent). These functions are called patch functions or assumed mode shapes. An approximation of the solution  $v(x,t)$  of (2.9) can be formed in  $H_N$  by

$$v_N(x,t) = \sum_{k=1}^N v_k(t) \psi_k(x) \quad (3.3)$$



i.e. assume separation of time and space variables with all spatial variation lumped into the patch functions  $\psi_k(x)$ . The choice of the coefficients  $v_k(t)$  remains; these are obtained by substitution of (3.3) into (2.9):

$$\frac{\partial v_N}{\partial t} = A v_N + Bf + E \quad (3.4)$$

where  $E$  is the equation error, and the  $v_k(t)$ 's are chosen so that

$$P_N(E) = 0 \quad (3.5)$$

This is called the Galerkin approximation; when it is carried out with the spline subspaces  $H_N$  described above, it produces the FEM approximation (3.3) where the coefficients  $v_k(t)$  are given by the entries of the solution vector  $v_N(t) = [v_1(t), \dots, v_N(t)]^T$  for the system of ordinary differential equations:

$$\tilde{M}_N \dot{v}_N = \tilde{A}_N v_N + \tilde{B}_N f \quad (3.6)$$

where  $\tilde{M}_N = [(\psi_\ell, \psi_k)]$ ,

$$\tilde{A}_N = [(\psi_\ell, A \psi_k)]$$

and  $\tilde{B}_N f = [(\psi_\ell, Bf)]$ .

The matrix  $\tilde{M}_N$  is symmetric and positive definite because  $\{\psi_k(x)\}_{k=1}^N$  are linearly independent:

$$z^T \tilde{M}_N z = \sum_{\ell=1}^N \sum_{k=1}^N z_\ell (\psi_\ell, \psi_k) z_k = \left( \sum_{\ell=1}^N z_\ell \psi_\ell, \sum_{k=1}^N z_k \psi_k \right) \geq 0 \text{ and}$$

$0 = z^T \tilde{M}_N z$  if and only if  $\sum_{k=1}^N z_k \psi_k = 0$  which leads to the  $z_k = 0$

due to the linear independence of the  $\psi_k$ . Therefore, (3.6) can be solved uniquely for  $v_N(t)$  whenever  $v_N(0)$  is specified, and hence the FEM approximation (3.3) is obtained. It is assumed that  $v_N(0)$  is given by the vector of coefficients of

$$v_N(0) = P_N v_0 \quad (3.7)$$

expanded in the basis  $\{\psi_k(x)\}_{k=1}^N$ . Note that

$$v_N = P_N v_N \quad (3.8)$$

The approximation (3.3) is called a semidiscretization of (2.9) because time  $t$  remains continuous in (3.6).

It should be noted that to obtain the most analytical benefit from the FEM, the approximation (3.3) should be obtained from the "weak" form of (2.9); however, we omit discussion of this technicality and refer the interested reader to [3].

### 3.2 Convergence of the Approximation

Let  $e(x,t) \equiv v(x,t) - v_N(x,t)$ , the FEM solution error, and consider from (2.9), (3.4), (3.5) and (3.8):

$$\left\{ \begin{array}{l} \frac{\partial v_N}{\partial t} = A_N v_N + B_N f, \quad v_N(0) = P_N v_0 \\ \frac{\partial e}{\partial t} = A_{RN} v_N + A e + B_R f, \quad e(0) = P_R v_0 \\ y = C_N v_N + C e \end{array} \right. \quad (3.9a)$$

$$(3.9b)$$

$$(3.9c)$$

where  $A_N = F_N A P_N$ ,  $B_N = P_N B$ ,  $C_N = C P_N$ ,  $B_R = P_R B$  and  $A_{RN} = P_R A P_N$ . Since  $P_N$  is orthogonal projection onto the finite dimensional subspace  $H_N$ ,  $(A_N, B_N, C_N)$  may be identified with appropriate matrices easily related to the ones in (3.6) by choosing the orthonormalized basis for  $H_N$  obtained from  $\{\psi_k\}_{k=1}^N$ ; in addition,  $A_{RN}$  is a bounded, linear operator. In general,  $A_{RN}$  is not zero, and hence  $H_N$  and  $H_R$  are not reducing subspaces for  $A$ ; furthermore,  $v_N \neq P_N v$ , and so  $v_N$  and  $e$  are not orthogonal to each other. Consequently, except for the special case where the exact modal data is known and  $H_N$  is chosen to be a modal subspace, the FEM

does not partition (2.9) into the previous form (2.13) analyzed in [17].

The FEM partition (3.9) must be analyzed separately.

From (3.9b) and the fact that  $A$  generates the  $C_0$  semigroup  $U(t)$ , obtain:

$$e(t) = U(t) P_R v_0 + \int_0^t U(t-\tau) [A_{RN} v_N(\tau) + B_R f(\tau)] d\tau \quad (3.10)$$

Take norms in (3.10) and use (2.11) to obtain:

$$\begin{aligned} \|e(t)\| \leq M_0 e^{-\epsilon t} (\|P_R v_0\| + \int_0^t e^{\epsilon \tau} [\|A_{RN} v(\tau)\| \\ + \|B_R\| \|f(\tau)\| + \|A_{RN}\| \|e(\tau)\|] d\tau) \end{aligned} \quad (3.11)$$

Now use Gronwall's Inequality ([25], p. 124) on (3.11) to obtain:

$$\begin{aligned} \|e(t)\| \leq M_0 e^{-\epsilon_N t} (\|P_R v_0\| + \int_0^t e^{\epsilon \tau} [\|A_{RN} v(\tau)\| \\ + \|B_R\| \|f(\tau)\|] d\tau) \end{aligned} \quad (3.12)$$

where  $\epsilon_N \equiv \epsilon - M_0 \|A_{RN}\|$ . This can be used to prove the following convergence result:

Theorem 3.1: The following two conditions guarantee convergence, over any finite interval of time, of the FEM approximation (3.3) to the actual open-loop solution of (2.9):

- (a)  $\|A_{RN}\| < \alpha$  where  $\alpha$  is independent of  $N$
- (b) The approximation is consistent, i.e.  $\lim_{N \rightarrow \infty} \|A_{RN} v\| = 0$  for all  $v$  in  $D(A)$ .

Furthermore, conditions (a) and (b) are equivalent to

$$\lim_{N \rightarrow \infty} \|A_{RN}\| = 0 \quad (3.13)$$

Proof: From [18], p. 151 lemma 3.5, we obtain that (a) and (b) imply (3.13) because  $D(A)$  is dense in  $H$ . Conversely, it is easy to see that (3.13) implies (a) and (b); so

these conditions are equivalent. Use (3.12) with  $\epsilon_N$  replaced by  $\epsilon - M_0 \alpha$  and let  $0 \leq t \leq T$ . From (3.2) and (3.13), obtain  $\lim_{N \rightarrow \infty} ||e(t)|| = 0$ . #

This is only a slight modification of the usual open-loop convergence results (e.g. [5] Sec. 4), but it will be convenient for our needs.

A priori estimates of the convergence rate of  $e(t)$  as a function of the mesh parameter  $h$  can often be made.

### 3.3 An Alternate Form of The FEM Approximation for Flexible Structures

We want to carry the development of the FEM approximation (3.3) and (3.6) a bit further. We can rewrite (3.3) as

$$v_N(x, t) = [u_N(x, t), w_N(x, t)]^T$$

where

$$\begin{cases} u_N(x, t) = \sum_{k=1}^N v_k(t) \psi_k^1(x) \\ w_N(x, t) = \sum_{k=1}^N v_k(t) \psi_k^2(x) \end{cases} \quad (3.14a)$$

$$(3.14b)$$

where  $\psi_k = [\psi_k^1 \ \psi_k^2]^T$  is in  $H_N$ . Now the entries in the matrices  $\tilde{M}_N$ ,  $\tilde{A}_N$ ,  $\tilde{B}_N$  in (3.6) are given by:

$$(v_\ell, \psi_k) = (\psi_\ell^1, A_0 \psi_k^1)_0 + (\psi_\ell^2, \psi_k^2) \quad (3.15a)$$

$$(v_\ell, A \psi_k) = (\psi_\ell^1, A_0 \psi_k^2)_0 - (\psi_\ell^2, A_0 \psi_k^1)_0 - b(\psi_\ell^2, \psi_k^2)_0 \quad (3.15b)$$

$$(v_\ell, Bf) = \sum_{i=1}^M (\psi_\ell^2, b_i)_0 f_i \quad (3.15c)$$

From this data (3.6) may be solved for the coefficients  $v_k(t)$  of the FEM approximation (3.3).

An alternate form of the FEM approximation is obtained by choosing spline subspaces  $S_N^0$  in  $H_0$  with  $\theta_1(x), \dots, \theta_N(x)$  as a basis for  $S_N^0$ .

Instead of making the approximation in the state space formulation (2.9), we do it directly on (2.1):

$$u_N(x,t) = \sum_{k=1}^N u_k(t) e_k(x) \quad (3.16)$$

where  $u_k(t)$  satisfy the system of ordinary differential equations:

$$M_N \ddot{\underline{u}}_N + b M_N \dot{\underline{u}}_N + A_N^0 \underline{u}_N = B_N^0 f \quad (3.17)$$

where  $\underline{u}_N = [u_1, \dots, u_N]^T$ ,  $M_N = [(\epsilon_\ell, \epsilon_k)_0]$ ,  $A_N^0 = [(\theta_\ell, A_0 \epsilon_k)_0]$ , and  $B_N^0 = [(\theta_\ell, b_i)]$ . It is easy to see that  $M_N$  and  $A_N^0$  are positive definite matrices because  $A_0$  is a self-adjoint, positive differential operator.

From the extended eigendata problem:

$$A_N^0 \underline{z}_k = \lambda_k \underline{z}_k \quad (3.18)$$

we obtain the approximate mode frequencies

$$\hat{\omega}_k = (\lambda_k)^{1/2} \quad (3.19a)$$

and approximate mode shapes:

$$\hat{e}_k(x) = \sum_{\ell=1}^N z_{k\ell} e_\ell(x) \quad (3.19b)$$

where  $\underline{z}_k = [z_{k1}, \dots, z_{kN}]^T$ . When we form the nonsingular matrix

$$V_N = [\underline{z}_1, \dots, \underline{z}_N] \quad (3.20)$$

we find that

$$V_N^T M_N V_N = I \quad (3.21a)$$

$$V_N^T A_N^0 V_N = \Lambda_N^0 = \text{diag} [\lambda_1, \dots, \lambda_N] \quad (3.21b)$$

The modal coordinates  $\underline{q}_N$  are defined by

$$\underline{u}_N = V_N \underline{q}_N \quad (3.22)$$

and (from (3.17) we obtain

$$\ddot{\underline{q}}_N + b \dot{\underline{q}}_N + \Lambda_N^0 \underline{q}_N = V_N^T B_N^0 f \quad (3.23)$$

This system of ordinary differential equations is decoupled into  $N$  second order systems and, hence, is quite easy to solve for  $\underline{q}_N$ . From (3.22), we can obtain the coefficients for the alternate approximation (3.16).

The alternate formulation (3.16) of the FEM for flexible structures is the standard one used in structural analysis; it is related to the one obtained in (3.3) or, equivalently, (3.14). Appropriate versions of (3.2) and (3.9) and a convergence result similar to Theo. 3.1 can be produced; for example, see [2] Sec. 7.3. For convenience we shall continue to use (3.3); however, alternate versions of our later results hold for (3.16).

#### 4.0 Feedback Controllers Associated with the FEM Approximation

The FEM reduced-order model associated with (2.9) is defined on  $H_N$  and given by

$$\begin{cases} \frac{\partial v_N}{\partial t} = A_N v_N + B_N f, & v_N(0) = P_N v_0 \\ y = C_N v_N \end{cases} \quad (4.1)$$

where  $(A_N, B_N, C_N)$  are defined in (3.9) and are determined by approximate modal data. Since  $H_N$  is a finite dimensional subspace,  $(A_N, B_N, C_N)$  may be identified with their matrices in an appropriate basis of  $H_N$ , and (4.1) is equivalent to a lumped parameter, state variable system for which a well developed feedback control theory exists, e.g. [23] or [24].

The controllability and observability of  $(A_N, B_N, C_N)$  are easily checked via simplified conditions obtained in [16]. These conditions are the same even though approximate, rather than exact, modal data are used; the conditions apply because of the low level of natural structural damping present in these structures (i.e.  $b$  is small in (2.1)).

The FEM Feedback Controller is based on the ROM (4.1) and defined by

$$\begin{cases} f = G_N \hat{v}_N \\ \dot{\hat{v}}_N = A_N \hat{v}_N + B_N f + K_N (y - \hat{y}) \\ \hat{y} = C_N \hat{v}_N ; \hat{v}_N(0) = 0 \end{cases} \quad (4.2)$$

where, due to the controllability and observability of the ROM, we can adjust the controller gains  $G_N$  and  $K_N$  so that  $A_N + B_N G_N$  and  $A_N - K_N C_N$  have any desired stability margin. The controller (4.2) has finite dimension  $N$  (where  $N = \dim H_N$ ); so we can abuse notation by using  $\dot{v}_N$  instead of  $\frac{\partial v_N}{\partial t}$ . This controller consists of a linear feedback control law and full order state estimator (full order in the sense that it is matched to the full order ROM). Much lower order controllers than (4.2) may be developed, but we leave that issue for another time.

We define the estimator error  $e_N \equiv \hat{v}_N - v_N$  and, from (3.9c) and (4.2), obtain:

$$\dot{e}_N = (A_N - K_N C_N) e_N + K_N C e \quad (4.3)$$

Also, from (3.9a) and (4.2), we have

$$\dot{v}_N = (A_N + B_N G_N) v_N + B_N G_N e_N \quad (4.4)$$

If there were no FEM solution error (i.e.  $e = 0$ ), then (4.3) and (4.4) would be designed with a desired stability margin. Consequently, the controller (4.2) would stabilize the model (4.1) by design; however, our principal interest is the closed-loop stability of the actual structure (2.9) with the controller (4.2).

### 5.0 Closed-Loop Stability: Main Results

The closed-loop system consists of the actual structure (2.9) with the FEM controller (4.2). The closed-loop system state is  $[v \ e_N]^T$  where  $v = v_N + e$  and  $e_N = \hat{v}_N - v_N$ ; equivalently, this state may be written

$$\omega = [v_N \ e_N \ e]^T$$

where  $\omega$  satisfies (from (3.9), (4.3) and (4.4)):

$$\frac{\partial \omega}{\partial t} = A_C \omega \quad (5.1)$$

where  $A_C = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  and

$$A_{11} = \begin{bmatrix} A_N + B_N G_N & B_N G_N \\ 0 & A_N - K_N C_N \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 \\ K_N C \end{bmatrix},$$

$$A_{21} = [A_{RN} + B_R G_N \quad B_R G_N] \text{ and } A_{22} = A$$

Since  $A_{11}$  represents a finite dimensional system, it is bounded and generates a  $C_0$  semigroup  $U_N(t)$  with the following growth property:

$$\|U_N(t)\| \leq M_N e^{-\epsilon_N t}, \quad t \geq 0 \quad (5.2)$$

where  $M_N \geq 1$  and  $\epsilon_N > 0$ ;  $\epsilon_N$  is the designed stability margin obtained in Sec. 4.0 by the choice of controller gains  $G_N$  and  $K_N$ . We have the following closed-loop stability result:

Theorem 5.1: If the FEM model  $(A_N, B_N, C_N)$  is controllable and observable for every  $N$  and if conditions (a) and (b) of Theo. 3.1 are satisfied, then controller gains  $G_N$  and  $K_N$  can be chosen so that  $A_C$  generates a  $C_0$  semigroup  $U_C(t)$  such that



$$\omega(t) = U_C(t) \omega(0) \quad (5.3)$$

and

$$\|U_C(t)\| \leq M_C e^{-\epsilon_C t}, \quad t \geq 0 \quad (5.4)$$

$$\text{where } \|\omega\| \equiv (\|v_N\|^2 + \|e_N\|^2 + \|e\|^2)^{1/2},$$

$$M_C = M_N M_0 \left(1 + \frac{\|K_N C\|}{\epsilon_N - \epsilon} + \frac{\|K_N C\|^2}{\epsilon_N - \epsilon}\right)^{1/2}$$

$$\epsilon_C = \epsilon - M_C \beta_N,$$

$$\beta_N = \|A_{RN}\| + 2\|B_R\| \|G_N\|, \text{ and } M_0, \epsilon \text{ are obtained from (2.11).}$$

The closed-loop system is stable, if  $G_N$  is bounded for all  $N$  and, for  $N$  sufficiently large,

$$M_C \beta_N < \epsilon \quad (5.5)$$

Proof: Choose  $G_N$  and  $K_N$  so that  $\epsilon_N > \epsilon$  since  $(A_N, B_N, C_N)$  controllable and observable. Let  $A_C = A_0 + \Delta A$  where

$$A_0 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ and } \Delta A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}$$

Note that  $A_0$  generates a  $C_0$  semigroup  $U_0(t)$  such that

$$\|U_0(t)\| \leq M_C e^{-\epsilon_0 t}, \quad t \geq 0$$

where  $\epsilon_0 \equiv \min(\epsilon_N, \epsilon) = \epsilon$ ; this follows from arguments similar to those used to prove Theo. 5.1 in [17]. Since  $\|\Delta A\| =$

$\|A_{21}\| \leq \beta_N$ ,  $\Delta A$  is a bounded perturbation of the semigroup generator  $A_0$ ; hence, by [25], Theo. 10.9, p. 210,  $A_C$  generates a  $C_0$  semigroup  $U_C(t)$  which satisfies (5.4). The conditions (a) and (b) of Theo. 3.1 imply that

$$\lim_{N \rightarrow \infty} \|A_{RN}\| = 0$$

and this, together with (3.2) and the boundedness of  $G_N$ , implies

$\lim_{N \rightarrow \infty} \beta_N = 0$ . If  $N$  can be chosen large enough to satisfy (5.5), then  $\epsilon_C > 0$  and the closed-loop system (5.3) is exponentially stable.

Note that  $M_C$  may increase as  $N \rightarrow \infty$  so it is not always certain that (5.5) can be satisfied even though  $\beta_N \rightarrow 0$ . \*

**Theorem 5.2:** In Theo. 5.1, if there exist positive constants  $\alpha_1$  and  $\alpha_2$  such that

$$M_C \leq \alpha_1 \beta_N^{\alpha_2 - 1} \quad (5.6)$$

then there is a finite integer  $N^*$  such that, for all  $N \geq N^*$ , (5.3) is exponentially stable.

**Proof:** This is a corollary to Theo. 5.1. From (5.6), we obtain

$M_C \beta_N \leq \alpha_1 \beta_N^{\alpha_2} \rightarrow 0$  as  $N \rightarrow \infty$ . Therefore, choose  $N^*$  such that  $\alpha_1 \beta_{N^*}^{\alpha_2} < \epsilon$  and, for all  $N \geq N^*$ , (5.5) is satisfied: the result follows from Theo. 5.1. \*

The above results provide conditions under which the closed-loop system (5.3) is stable and the FEM solution error  $e(t)$  satisfies

$$\|e(t)\| \leq \|\omega(t)\| \leq 2 M_C e^{-\epsilon_C t} \|v_0\|, \quad t \geq 0 \quad (5.7)$$

because

$$\|\omega(0)\|^2 = 2 \|P_N v_0\|^2 + \|P_R v_0\|^2 \leq 2 \|v_0\|^2.$$

This is an a priori convergence estimate for the closed-loop system.

Since  $e = v - v_N = (v - P_N v) + (P_N v - v_N)$  and  $v - P_N v = P_R v \rightarrow 0$  as  $N \rightarrow \infty$  by (3.2), we may be interested in the behavior of the term  $P_N v - v_N$  as  $N \rightarrow \infty$ . However, from the properties of the Galerkin projection  $P_N$ ,

$$\begin{cases} P_N e = P_N P_R v + P_N (P_N v - v_N) = P_N v - v_N & (5.8a) \\ P_R e = P_R P_R v + P_R (P_N v - v_N) = P_R v & (5.8b) \end{cases}$$

i.e.,  $P_N v - v_N = P_N e$  and  $v - P_N v = P_R e$  form the orthogonal components of  $e$  in  $H_N$  and  $H_N^\perp$ , respectively. From (3.9b) and (4.2), obtain:

$$\begin{cases} \frac{\partial P_N e}{\partial t} = A_N P_N e + A_{NR} v_R & (5.9a) \\ \frac{\partial v_R}{\partial t} = A_{RN} P_N e + A_R v_R + (A_{RN} + B_R G_N) v_N + B_R G_N e_N & (5.9b) \end{cases}$$

where  $v_R \equiv P_R v$ ,  $A_{NR} \equiv P_N A P_R$ , etc. Also, (4.3) becomes

$$\dot{e}_N = (A_N - K_N C_N) e_N + K_N C_N P_N e + K_N C_R v_R \quad (5.10)$$

Let  $\omega = [v_N, e_N, P_N e, v_R]^T$  be the closed-loop system state, and, from (4.4), (5.9) and (5.10), obtain

$$\frac{\partial \omega}{\partial t} = Q_C \omega \quad (5.11)$$

where  $Q_C = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$  and

$$Q_{11} = \begin{bmatrix} A_N + B_N G_N & B_N G_N & 0 \\ 0 & A_N - K_N C_N & K_N C_N \\ 0 & 0 & A_N \end{bmatrix}, \quad Q_{12} = \begin{bmatrix} 0 \\ K_N C_R \\ A_{NR} \end{bmatrix}$$

$$Q_{21} = [A_{RN} + B_R G_N \quad B_R G_N \quad A_{RN}], \quad \text{and} \quad Q_{22} = A_R$$

The following alternate closed-loop stability result may prove useful in those cases where conditions (a) and (b) of Theo. 5.1 are not satisfied by the FEM approximation:

**Theorem 5.3:** Assume, for every  $N$ ,

- (1) the ROM  $(A_N, B_N, C_N)$  is controllable and observable
- (2)  $A_N$  is (exponentially) stable with margin  $\sigma_N$
- (3)  $A_R$  generates an exponentially stable  $C_0$  semigroup  $U_R(t)$  with
 
$$||U_R(t)|| \leq M_R e^{-\sigma_R t}, \quad t \geq 0$$
- (4)  $\lim_{N \rightarrow \infty} ||A_{NR}|| = 0$

Then controller gains  $G_N$  and  $K_N$  can be chosen so that  $Q_C$  in (5.11) generates a  $C_0$  semigroup  $U_C(t)$  with

$$||U_C(t)|| \leq M_C e^{-\epsilon_C t}, \quad t \geq 0 \quad (5.12)$$

where  $M_C = \max(M_{11}, M_R)$ ,

$$\epsilon_C = \epsilon_0 - M_C \Gamma_N, \quad \epsilon_0 = \min(\sigma_N, \sigma_R),$$

$$\Gamma_N = ||K_N|| ||C_R|| + ||A_{NR}||$$

and  $Q_{11}$  generates the  $C_0$  semigroup  $U_{11}(t)$  with

$$||U_{11}(t)|| \leq M_{11} e^{-\sigma_N t}, \quad t \geq 0.$$

If  $K_N$  is bounded for all  $N$  and, for  $N$  sufficiently large,

$$M_C \Gamma_N < \epsilon_0 \quad (5.13)$$

the closed-loop system (5.11) is exponentially stable.

The proof of this result is sufficiently similar to that of Theo. 5.1 that we shall omit it. In some cases, the conditions (1)-(4) of Theo.

5.3 may be easier to verify than those of Theo. 5.1. Again, the stability of (5.12) leads to an a priori convergence bound on the FEM solution error:

$$||e(t)|| \leq 2M_C e^{-\epsilon_C t} ||v_0||, \quad t \geq 0 \quad (5.14)$$

## 6.0 Conclusions

The use of the finite element method (FEM) for numerical simulation of complex distributed parameter systems is well known. In this paper the use of the FEM to produce reduced-order models and finite dimensional feedback controllers for distributed parameter systems has been investigated with emphasis on application to control of large aerospace structures. Our principal concern has been to determine the stability of the controller in closed-loop with the actual distributed parameter system even though the controller is synthesized using a FEM approximation. Our main results, Theorems 5.1-5.3, give conditions under which stable closed-loop operation can be expected. Such conditions are especially desirable for large aerospace structure control because the FEM model is the tool most widely accepted for structural analysis and large scale computer codes already exist to produce such models for very complex structures.

Acknowledgments:

The author wishes to acknowledge the support of this work by the National Aeronautics and Space Administration Contract NAS 9-16053.

# References:

1. Satellite Power System: Concept Development and Evaluation, U.S. Dept. of Energy and Nat'l Aeronautics and Space Administration Report, DOE/ER-0023, 1978.
2. G. Strang and G. Fix, An Analysis of the Finite Element Method, Prentice-Hall, NJ, 1973.
3. J. Oden and J. Reddy, An Introduction to the Mathematical Theory of Finite Elements, J. Wiley and Sons, NY, 1976.
4. J. Oliger, Methods for Time-Dependent Partial Differential Equations, in Numerical Analysis - Proc. of Symposia in Appl. Math., Vol. 22, G. Golub and J. Oliger, eds., Am. Math. Soc., RI., 1978.
5. D. Gottlieb and S. Orszag, Numerical Analysis of Spectral Methods: Theory and Application, Soc. for Ind. and Appl. Math., PA, 1977.
6. P. Orner, P. Salamon and W. Yu, Least Squares Simulation of Distributed Systems, IEEE Trans. Autom. Contr., AC-20(1975), 75-83.
7. G. Knowles, Some Problems in the Control of Distributed Systems and Their Numerical Solution, SIAM J. Contr. and Opt., 17(1979), 5-22.
8. I. Lasiecka, Boundary Control of Parabolic Systems: Finite Element Approximation, Appl. Math. Opt., 6(1980), 31-62.
9. I. Lasiecka, Unified Theory for Abstract Parabolic Boundary Problems - A Semigroup Approach, Appl. Math. Opt. (to appear).
10. S. Prabhu and I. McCausland, Optimal Control of Linear Diffusion Processes with Quadratic Error Criteria, Automatica, 8(1972), 299-308.
11. R. Gran, A Survey of the Large Structures Control Problem, Proc. of IEEE Conf. on Decision and Control, Ft. Lauderdale, FL, 1979.
12. L. Meirovitch and H. Öz, An Assessment of Methods for the Control of Large Space Structures, Proc. of Joint Autom. Contr. Conf., Denver, CO, 1979.
13. M. Balas, Some Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams, IEEE Trans. Autom. Contr. (to appear).
14. W. Flannelly and G. Lang, Modal Analysis for Managers, Sound and Vibration, November 1979.

15. L. Meirovitch and H. Öz, Control of Large-Order Discretized Systems, 13th Asilomar Conference on Circuits, Systems and Computers, Pacific Grove, CA, 1979.
16. M. Balas, Feedback Control of Flexible Systems, IEEE Trans. Autom. Contr., AC-23(1978), 673-679.
17. M. Balas, Feedback Control of Dissipative Hyperbolic Distributed Parameter Systems with Finite Dimensional Controllers, submitted to SIAM J. Contr. and Opt.
18. T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, NY, 1966.
19. J. Goldstein, Semigroups of Operators and Applications, Tulane University, New Orleans, LA, Math. Lecture Notes, 1979.
20. J. Goldstein and S. Rosencrans, Energy Decay and Partition for Dissipative Wave Equations, J. Differential Equations (to appear).
21. M. Balas, Control of Flexible Structures in the Presence of Certain Nonlinear Disturbances, Proc. of Joint Autom. Contr. Conf., Denver, CO, 1979.
22. M. Balas, Modal Control of Certain Flexible Dynamic Systems, SIAM J. Contr. and Opt., 16(1978), 450-462.
23. H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, J. Wiley and Sons, NY, 1972.
24. D. Russell, Mathematics of Finite Dimensional Control Systems: Theory and Design, M. Dekker, NY, 1979.
25. R. Curtain and A. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic Press, NY, 1977.



C. REDUCED-ORDER FEEDBACK CONTROL OF DISTRIBUTED PARAMETER  
SYSTEMS VIA SINGULAR PERTURBATIONS METHODS

Mark J. Balas  
Associate Professor  
Electrical and Systems Engineering Department  
Rensselaer Polytechnic Institute  
Troy, New York 12181

## ABSTRACT

Implementable feedback control of distributed parameter systems must often be based on reduced-order models due to the infinite dimensional nature of the actual open-loop system. In this paper, we analyze the behavior of controllers designed via reduced-order models obtained with singular perturbations techniques. When such controllers are used in the actual distributed parameter system, the closed-loop stability is in question. The results presented here provide bounds on the smallness of the singular perturbations parameter to ensure stable operation; such a priori bounds may be used to evaluate potential reduced-order controllers for distributed parameter systems.

## 1. INTRODUCTION

Distributed Parameter Systems (DPS) are described by partial differential equations. Examples of DPS include heat diffusion and chemical processes, wave propagation, and mechanically flexible structures, such as large flexible spacecraft and satellites, high-speed aircraft and surface vehicles, and large civil engineering structures, e.g., bridges and tall buildings.

The state spaces for these systems have infinite dimension. Thus, it is impractical or impossible to implement feedback controllers based on complete models of these systems; hence, reduced-order models must be used for the controller design. A great variety of reduced-order modeling techniques exist for general systems, e.g., the bibliography of [1], and new techniques are presently being developed in the specialized areas like large aerospace structures, e.g., [2], [3]. Of particular interest are the model reduction techniques based on asymptotic methods, such as multiple time scales and singular perturbations, e.g., [4]-[7] and the excellent survey [8]. These papers address the large scale, but finite dimensional or lumped parameter, systems; to our knowledge, very little has been done with asymptotic methods for DPS with the exceptions: [9] Chapt. 5, [10], and the survey [11].

Our interest here is less in the area of derivation of reduced-order models for DPS (although that is an important topic in its own right); rather, we are concerned with the successful operation of the controller, designed on a reduced-order model, in closed-loop with the full system. Other work on this basic topic may be found, for example, in [2], [12]-[15], and, for singular perturbations methods, [16]-[24]; none of these singular perturbations papers deals with DPS, with the exception of [24]. This paper

extends the preliminary results on singular perturbations and stability for DPS obtained in [24] and obtains an upper bound on the singular parameter for stable, closed-loop operation. Our results for DPS are in the same spirit as those of [21]-[23] for lumped parameter systems; the method of proof is different due to the infinite dimensional nature of the DPS problem.

We consider linear, time-invariant distributed parameter systems with the following form:

$$\begin{cases} v_t = Av + Bf & , & v(0) = v_0 \\ y = Cv + Df \end{cases} \quad (1.1)$$

where the state  $v$  is in a Hilbert space  $H$  with inner product  $(\dots)$  and associated norm  $||\cdot||$  and the control vector  $f$  and observation vector  $y$  have dimensions  $M$  and  $P$ , respectively, which denote the number of (independent) actuators and sensors. When the system (1.1) is a distributed parameter system (DPS), the  $\dim H = \infty$  and the operator  $A$  is (usually) an unbounded, differential operator with domain  $D(A)$ , containing all states which satisfy the boundary conditions of the problem & dense in  $H$ , while the input and output operators  $B$  and  $C$  have finite rank  $M$  and  $P$ , respectively, and  $D$  is a  $P \times M$  matrix.

The operator  $A$  generates a  $C_0$  semigroup  $U(t)$  on  $H$ . This semigroup  $U(t)$  is usually exponentially stable, i.e., it has the growth property:

$$||U(t)|| \leq K_0 e^{-\delta_0 t} \quad , \quad t \geq 0 \quad (1.2)$$

where  $K_0 \geq 1$  and  $\delta_0 > 0$ . In physical systems of "hyperbolic"-type, energy dissipative mechanisms make (1.2) true, even for DPS; however,  $\delta_0$  may be quite small, as it often is in the case of large aerospace structures [2].

In the following sections, we develop the general idea of reduced-order models for the systems of type (1.1) and obtain results on the operation

of controllers, based on these reduced-order models, in closed-loop with the actual system (1.1).

## 2. REDUCED-ORDER MODELING AND CONTROLLER DESIGN

Let  $H_N$  and  $H_R$  be subspaces of the total state space  $H$  with  $\dim H_N = N < \infty$  and  $H = H_N \oplus H_R$ . Define the projection operators  $P_N$  and  $P_R$  (not necessarily orthogonal) and let  $v_N = P_N v$  and  $v_R = P_R v$ . This decomposes  $v$  into  $v = v_N + v_R$  and the system (1.1) into

$$\dot{v}_N = A_N v_N + A_{NR} v_R + B_N f, \quad v_N(0) = P_N v_0 \quad (2.1)$$

$$\dot{v}_R = A_{RN} v_N + A_R v_R + B_R f, \quad v_R(0) = P_R v_0 \quad (2.2)$$

$$y = C_N v_N + C_R v_R + D f \quad (2.3)$$

Note that all parameters,  $A_N$ ,  $A_{NR}$ , etc., with the exception of  $A_R$ , are bounded operators because they involve projection onto the finite dimensional subspace  $H_N$ . We abuse notation slightly by writing  $\dot{v}_R$  for  $\partial v_R / \partial t$ .

Henceforth, we assume that  $A_R$  generates a semigroup  $U_R(t)$  on  $H_R$  which satisfies the growth condition:

$$\|U_R(t)\| \leq K_R e^{-\delta_R t}, \quad t \geq 0 \quad (2.4)$$

where  $K_R \geq 1$  and  $\delta_R > 0$ . In the special case of reducing subspaces (i.e., both  $H_N$  and  $H_R$  are  $A$ -invariant),  $\delta_R = \delta_0$  and  $K_R = \|P_R\| K_0$  where  $\delta_0$ ,  $K_0$  are as given in (1.2). In general,  $K_R$  and  $\delta_R$  depend on the choice of  $H_N$ ,  $H_R$  subspaces. The growth condition (2.4) will hold for "parabolic", as well as "hyperbolic" systems even though the more stringent (1.2) does not hold.

The reduced-order model (ROM) for the system is (2.1) and (2.3) with  $C_R$  and  $A_{NR}$  assumed to be zero:

$$\begin{cases} \dot{\tilde{v}}_N = A_N \tilde{v}_N + B_N f, & \tilde{v}_N(0) = P_N v_0 \\ \tilde{y} = C_N \tilde{v}_N + D f \end{cases} \quad (2.5)$$

Thus, the ROM depends on what choice of subspace  $H_N$  is made and what type of projection  $P_N$  is used (or alternatively, what  $H_R$  is). The subspace  $H_N$  is called the ROM subspace and the subspace  $H_R$ , the residuals subspace. The terms  $A_{NR} v_R$  and  $A_{RN} v_N$  are called model error and  $B_R f$  and  $C_R v_R$  are called control and observation spillover, respectively.

### 3. REDUCED-ORDER MODELING AND CONTROLLER DESIGN VIA SINGULAR PERTURBATIONS

For certain choices of subspaces  $H_N$ ,  $H_R$  in Sec. 2, it is possible to produce ROM's such that (2.1)-(2.3) become:

$$\dot{v}_N = A_N v_N + A_{NR} v_R + B_N f \quad (3.1)$$

$$\epsilon \dot{v}_R = A_{RN} v_N + A_R v_R + B_R f \quad (3.2)$$

$$y = C_N v_N + C_R v_R + D f \quad (3.3)$$

where the singular perturbation parameter  $\epsilon \geq 0$  represents some small parameter dependence, such as electrical networks with parasitics or dynamical systems with small masses and time constants (see [25]); alternatively,  $\epsilon$  may represent a ratio of time-scales in the system (see [7]). This approach is especially valuable when the ROM is based on vibration modes for a mechanically flexible structure and the frequency separation of slow and fast modes can be used to provide the decomposition of (3.1)-(3.3). A survey of singular perturbation techniques for model reduction is presented in [8]. It should be noted that one of the most difficult tasks may be to formulate a particular DPS problem into the singular perturbations format (3.1)-(3.3).

The parameters such as  $A_N$  may involve regular perturbations in  $\epsilon$ , as well, i.e.,

$$A_N = A_N^0 + \epsilon \tilde{A}_N^\epsilon \quad (3.4)$$

where  $\tilde{A}_N^\epsilon$  is bounded for all  $\epsilon$  and  $\tilde{A}_N^0 = 0$ . Also, it is possible to have multi-

parameter singular perturbations where several  $\epsilon$  parameters are present; see e.g., [26]-[27]. To simplify the presentation, we shall not consider either of the above situations here.

The singular perturbations ROM is obtained when  $\epsilon = 0$  and is given by

$$\begin{cases} \dot{\tilde{v}}_N = \bar{A}_N \tilde{v}_N + \bar{B}_N f & , \quad \tilde{v}_N(0) = P_N v_0 \\ \hat{y} = \bar{C}_N \tilde{v}_N + \bar{D} f \end{cases} \quad (3.5)$$

where  $\bar{A}_N = A_N - A_{NR} A_R^{-1} A_{RN}$ ,  $\bar{B}_N = B_N - A_{NR} A_R^{-1} B_R$ ,  $\bar{C}_N = C_N - C_R A_R^{-1} A_{RN}$ , and  $\bar{D} = D - C_R A_R^{-1} B_R$ . This is somewhat different from the usual ROM (2.5) due to the parameter correction terms involving  $A_R^{-1}$ . In the special case of reducing subspaces  $H_N$ ,  $H_R$  (as in [24]), the model error terms  $A_{NR}$  and  $A_{RN}$  are zero, and  $A_N = A_N$ ,  $\bar{B}_N = B_N$ , and  $\bar{C}_N = C_N$ ; however,  $\bar{D} \neq D$ . Therefore, even in this special case the ROM (3.5) is not quite the same as the usual ROM (2.5).

Note that the ROM (3.5) is finite dimensional ( $\dim v_N = N$ ) and the parameters  $(\bar{A}_N, \bar{B}_N, \bar{C}_N, \bar{D})$  may be identified with their corresponding matrices in an appropriate basis for  $H_N$ . Henceforth, we assume the ROM (3.5) is controllable and observable; for any ROM, this is easy to verify with the usual rank tests for finite dimensional systems ([28] Chapt. 11).

Also, note that, although  $A_R$  is usually not bounded for a DPS,  $A_R^{-1}$  is bounded due to (2.4). In fact, from [28], Theo. 8.9,

$$\|A_R^{-1}\| \leq \frac{K_R}{\delta_R} \quad (3.6)$$

The reduced-order controller based on the ROM (3.5) is given by

$$\begin{cases} f = \bar{G}_N z \\ \dot{z} = \bar{A}_N z + \bar{B}_N f + \bar{K}_N(y - \hat{y}) & , \quad z(0) = 0 \\ \hat{y} = \bar{C}_N z + \bar{D} f \end{cases} \quad (3.7)$$

where  $\dim z = N$ . This finite dimensional controller can be implemented for the DPS, and the gains  $\bar{G}_N$ ,  $\bar{K}_N$  can be designed for closed-loop stability when  $\epsilon = 0$ ; however, successful closed-loop operation is in question when  $\epsilon > 0$ .

#### 4. MAIN RESULTS: CLOSED-LOOP STABILITY

The fundamental question is whether a stable closed-loop reduced-order system ( $\epsilon = 0$ ) will remain stable when the same controller (3.9) is used with the actual system (3.1)-(3.3) when  $\epsilon > 0$  (although small). The answer, as given below, is that it will when  $\epsilon$  is small enough; bounds on the smallness of  $\epsilon$  are obtained.

The closed-loop behavior of the total system (3.1)-(3.3) with the controller (3.7) is governed by

$$\begin{bmatrix} w_1 \\ \epsilon w_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (4.1)$$

where  $w_1 = [v_N^T e_N^T]^T$ ,  $e_N = z - v_N$ ,  $w_2 = v_R$ , and

$$H_{11} = \begin{bmatrix} A_N + B_N \bar{G}_N & B_N \bar{G}_N \\ Q_N & A_N - \bar{K}_N C_N + Q_N \end{bmatrix}$$

$$Q_N = (\bar{K}_N C_R - A_{NR}) A_R^{-1} (A_{RN} + B_R \bar{G}_N)$$

$$H_{12} = \begin{bmatrix} A_{NR} \\ \bar{K}_N C_R - A_{NR} \end{bmatrix}$$

$$H_{21} = [A_{RN} + B_R \bar{G}_N \quad B_R \bar{G}_N]$$

and  $H_{22} = A_R$ . Note that  $H_{11}$ ,  $H_{12}$ ,  $H_{21}$  are all bounded operators, and  $H_{22}^{-1}$  is bounded by (3.6); therefore let



$$\begin{aligned} ||H_{11}|| \leq M_1, \quad ||H_{12}|| \leq M_2, \quad ||H_{21}|| \leq M_3, \\ ||H_{22}^{-1}|| \leq M_4 \equiv K_R/\delta_R, \quad ||H_{22}^{-1} H_{21}|| \leq M_5 \end{aligned} \quad (4.2)$$

When  $\epsilon = 0$ , (3.1) reduces to

$$\dot{\tilde{w}}_1 = H_1 \tilde{w}_1 \quad (4.3)$$

which represents the reduced-order design consisting of the ROM (3.5) and the reduced-order controller (3.7) in closed-loop. The reduced-order closed-loop operator is given by:

$$H_1 = H_{11} - H_{12} H_{22}^{-1} H_{21} = \begin{bmatrix} \bar{A}_N + \bar{B}_N \bar{G}_N & \bar{B}_N \bar{G}_N \\ 0 & \bar{A}_N - \bar{K}_N \bar{C}_N \end{bmatrix}$$

which can be made stable by the choice of gains  $\bar{G}_N, \bar{K}_N$ , since the ROM (3.5) is assumed controllable and observable in Sec. 3. Therefore, let the transition matrix  $U_1(t)$  associated with  $H_1$  satisfy:

$$||U_1(t)|| \leq K_1 e^{-\delta_1 t}, \quad t \geq 0 \quad (4.4)$$

Furthermore,  $H_1$  is bounded:

$$||H_1|| \leq M_6 \quad (4.5)$$

where

$$M_6 = M_1 + M_2 M_5 \leq K_1/\delta_1$$

When  $\epsilon > 0$ , we can write (4.1) as

$$\dot{w} = H w = \begin{bmatrix} H_{11} & H_{12} \\ H_{21}/\epsilon & H_{22}/\epsilon \end{bmatrix} w \quad (4.6)$$

where  $w = [w_1^T w_2^T]^T$ . The primary question to be answered in this section is whether the semigroup  $U_c(t)$  generated by  $H$  in (4.6) remains stable for small, positive  $\epsilon$ , i.e.,

$$||U_c(t)|| \leq K_c e^{-\delta_c t}, \quad t \geq 0 \quad (4.7)$$

The answer is that it does remain stable and the following theorem -- our main result -- gives a bound on the smallness required of  $\epsilon$ :

**THEOREM 1:** Let the controller gains  $\bar{G}_N, \bar{K}_N$  be chosen so that  $H_1$  in (4.3) has stability margin  $\delta_1$  as in (4.4). There exists an  $\epsilon_0 > 0$  such that, for all  $0 < \epsilon \leq \epsilon_0$ , the closed-loop system (4.6) -- consisting of the full-order DPS (1.1) and the reduced-order controller (3.7) based on the ROM (3.5) -- is (exponentially) stable and the controller state  $z$  converges (exponentially) to the reduced state  $v_N$ ; this means (4.7) is satisfied. An upper bound for  $\epsilon_0$  is given by:

$$\epsilon_0 < \frac{\delta_R}{(1 + M_5) K_R M_2 (M_2 + M_6)} \cdot \min \left( \frac{\delta_1}{K_1}, \gamma M_2 \right) \quad (4.8)$$

where  $\gamma$  is defined later in Lemma 1, and  $K_c, \delta_c$ , in (4.7), are given by:

$$\delta_c = \min (\hat{\delta}_1, \hat{\delta}_2/\epsilon_0) \quad (4.9)$$

$$K_c = K_1 K_R (1 + \alpha + \alpha^2)^{1/2} (3 + 3 M_5 + M_5^2) \quad (4.10)$$

where

$$\hat{\delta}_1 \equiv \delta_1 - \epsilon_0 K_1 M_2 M_4 (1 + M_5) (M_2 + M_6)$$

$$\hat{\delta}_2 \equiv \delta_R - \epsilon_0 K_R M_2 (1 + M_5)$$

$$\alpha \equiv M_2/|\Delta|, \quad \Delta \equiv \hat{\delta}_1 - \hat{\delta}_2/\epsilon_0,$$

and the constants are obtained from (2.4), (4.2), (4.4), and (4.5).

The proof of Theo. 1 requires the following two lemmata:

**LEMMA 1:** There exists  $\epsilon_1 > 0$  such that, for all  $0 < \epsilon \leq \epsilon_1$ , the nonlinear mapping  $h(L) = H_{22}^{-1} H_{21} + \epsilon H_{22}^{-1} L (H_{11} - H_{12} L)$  defined on  $\Omega = \{L \mid \|L - H_{22}^{-1} H_{21}\| < 1\}$  has a unique fixed-point  $L^* = L^*(\epsilon)$

in  $\Omega$  (hence,  $||L^*|| < 1 + M_5$ ). Furthermore,  $z_2 = w_2 + L^* w_1$  transforms (4.1) into

$$\begin{bmatrix} \dot{w}_1 \\ \epsilon z_2 \end{bmatrix} = \begin{bmatrix} H_{11} - \epsilon W_1 & H_{12} \\ 0 & H_{22} + \epsilon L^* H_{12} \end{bmatrix} \begin{bmatrix} w_1 \\ z_2 \end{bmatrix} \quad (4.11)$$

where  $W_1 \equiv H_{12} H_{22}^{-1} L^* (H_{11} - H_{12} L^*)$ . An upper bound for  $\epsilon_1$  is  $[M_4 (1 + M_5) (M_2 + M_6)]^{-1}$  and we write  $\epsilon_1 = \gamma [M_4 (1 + M_5) (M_2 + M_6)]^{-1}$  where  $0 < \gamma < 1$ .

**LEMMA 2:** There exists  $\epsilon_0$  such that, for all  $0 < \epsilon \leq \epsilon_0$ ,

$$\begin{aligned} [||w_1(t)||^2 + ||z_2(t)||^2]^{1/2} &\leq \tilde{K}_c e^{-\delta_c t} \\ [||w_1(0)||^2 + ||z_2(0)||^2]^{1/2} &\end{aligned} \quad (4.12)$$

where  $\tilde{K}_c = K_1 K_R (1 + \alpha + \alpha^2)^{1/2}$  and  $\alpha, \delta_c$  the same as in Theo. 1.

An upper bound for  $\epsilon_0$  is given by (4.8).

The proof of Lemma 1 involves a contraction mapping argument ([29], Theo. II. 1.1, p. 24); however, the results of [30] might be used to obtain a different (possibly, larger) bound for  $\epsilon_1$ . It is well known ([29], p. 24) that

$$L^* = \lim L_k$$

where  $L_k$  is obtained via the algorithm:

$$L_{k+1} = h(L_k), \quad k = 0, 1, 2, \dots, \quad (4.13)$$

and  $L_0$  any member of  $\Omega$ . This algorithm may be implemented to calculate the desired  $L^*$ . The following gives an indication of the convergence rate:

$$||L_k - L^*|| \leq \frac{\gamma^k}{1-\gamma} ||L_0 - L^*|| \quad (4.14)$$

where  $\gamma$  is defined in Lemma 1.

ORIGINAL PAGE IS  
OF POOR QUALITY

The proof of Lemma 2, and hence Theo. 1, makes use of Lemma 1 and results from [28]. The proofs of Lemma 1 and 2 and Theo. 1 appear in Appendices I-III.

## 5. CONCLUSIONS

Our main result (Theo. 1) provides an upper bound (4.8) on the smallness of the singular perturbations parameter ( $\epsilon$ ) which ensures stable closed-loop operation of a finite dimensional controller with the full-order distributed parameter system. Since the term  $H_{12}$  and  $H_{21}$  in (4.6) appear because of spillover and model error, the bounds (and hence, the stability) improve when these terms can be reduced (i.e., when  $M_2$  and  $M_3$  can be made smaller). Lemma 1&2 taken together form an infinite dimensional version of the Klimushchev-Krasovskii result [8].

When the distributed parameter system can be put into the singular perturbations format, (3.1)-(3.3), the stability condition presented here can be checked with only a limited knowledge of the unmodeled residuals present in the full-order system. This makes it possible to synthesize low-order controllers for distributed parameter systems via general reduced-order modeling techniques and analyze their operation in closed-loop with the actual system. Such a result appears to be particularly applicable to distributed parameter systems with multiple time-scales or high-low frequency separation, as is often the case in large aerospace structures [2], [31]. The general modeling issue, i.e. obtaining a singular perturbations format, for large-scale or distributed parameter systems is quite complex; see [32] for further discussion.

## Acknowledgements

I would like to acknowledge helpful conversations on this work with S.H. Javid (Systems Control Inc.) and encouragement from P. Kokotović (University of Illinois).

This work was partially supported by NASA Contract NAS 9-16053.

# REFERENCES

1. R. Genesio, and M. Milanese, A Note on the Derivation and Use of Reduced-Order Models, IEEE Trans. Autom. Contr., AC-21 (1976), 118-122.
2. M. Balas, Some Trends in Large Space Structure Control Theory: Fondest Hopes; Wildest Dreams, Invited Survey for Joint Autom. Contr. Conf., Denver, CO, 1979.
3. R. Skelton, Modal Cost Analysis in Model and Controller Reduction, 17th Allerton Conf. on Comm., Contr., and Computing, Monticello, IL, 1979.
4. P. Kokotović, and W. Perkins, eds., Singular Perturbations - Order Reduction in Control System Design, A Symp. of the Am. Autom. Contr. Council, 1972 Joint Autom. Contr. Conf., Stanford, CA 1972.
5. P. Kokotović, and P. Sannuti, Singular Perturbation Method for Reducing the Model Order in Optimal Control Design, IEEE Trans. Autom. Contr., AC-13 (1968), 377-384.
6. R. Ramnath, and G. Sundri, A Generalized Multiple Time Scales Approach to a Class of Linear Differential Equations, J. Math. Anal. and Appl., 28 (1969).
7. P. Kokotović, J. Allemong, J. Winkelman, and J. Chow, Singular Perturbation and Iterative Separation of Time Scales, Automatica, 16 (1980), 23-33.
8. P. Kokotović, R. O'Malley, and P. Sannuti, Singular Perturbations and Order Reduction in Control Theory - An Overview, Automatica, 12 (1976), 123-132.
9. J. L. Lions, Some Aspects of the Optimal Control of Distributed Parameter Systems, SIAM, Philadelphia, PA, 1972.
10. J. L. Lions, Remarks on the Theory of Optimal Control of Distributed Systems, Chapt. 3, in Control Theory of Systems Governed by Partial Differential Equations, A. Aziz, et.al., eds., Academic Press, NY, 1977.
11. J. L. Lions, Asymptotic Methods in the Optimal Control of Distributed Systems, Automatica, 14 (1978), 199-211.
12. D. D. Šiljak, Large Scale Dynamic Systems: Stability and Structure, Elsevier-North Holland Inc., NY, 1977.
13. A. N. Michel, and R. K. Miller, Qualitative Analysis of Large Scale Dynamical Systems, Academic Press, NY, 1977.
14. M. Balas, Feedback Control of Dissipative Distributed Parameter Systems, IEEE Conf. on Control and Decision, Ft. Lauderdale, FL, 1979.

15. M. Balas, Feedback Control of Wave Propagation in Symmetric Hyperbolic PDE Systems, 13th Asilomar Conf., Pacific Grove, CA, 1979.
16. M. Suzuki, and M. Miura, Stabilizing Feedback Controllers for Singularly Perturbed Linear Constant Systems, IEEE Trans. Autom. Contr., AC-21 (1976), 123-124.
17. K. D. Young, Feedback Design of Singularly Perturbed Systems, 17th Allerton Conf. on Comm., Contr., and Computing, Monticello, IL, 1979.
18. S. Campbell, On a Singularly Perturbed Autonomous Linear Control Problem, IEEE Trans. Autom. Contr., AC-24 (1979), 115-117.
19. N. Sandell, Robust Stability of Systems with Application to Singular Perturbations, Automatica, 15 (1979), 467-470.
20. J. Sesak, P. Likins, and T. Coradetti, Flexible Spacecraft Control by Model Error Sensitivity Suppression, J. of Astronautical Sciences, 27 (1979), 131-156.
21. L. Zien, An Upper Bound for the Singular Parameter in a Stable Singularly Perturbed System, J. Franklin Inst., 295 (1973), 373-381.
22. S. H. Javid, Uniform Asymptotic Stability of Linear Time-Varying Singularly Perturbed Systems, J. Franklin Inst., 305 (1978), 27-37.
23. S. H. Javid, Stabilization of Time-Varying Singularly Perturbed Systems by Observer-Based Slow-State Feedback, (to appear).
24. M. Balas, Singular Perturbations and Model Reduction Methods for Large Scale and Distributed Parameter Systems, 17th Allerton Conf., Monticello, IL, 1979.
25. P. Kokotović, A Control Engineer's Introduction to Singular Perturbations, contained in Ref. 4.
26. H. Khalil, and P. Kokotović, Control of Linear Systems with Multiparameter Singular Perturbations, Automatica, 15 (1979), 197-207.
27. H. Khalil, and P. Kokotović, D-Stability and Multiparameter Singular Perturbation, SIAM J. Contr. and Opt., 17 (1979), 56-65.
28. R. Curtain and A. Pritchard, Functional Analysis in Modern Applied Mathematics, Academic Press, NY, 1977.
29. J. Holtzman, Nonlinear System Theory: A Functional Analysis Approach, Prentice-Hall, NJ, 1970.
30. P. Kokotović, A Riccati Equation for Block-Diagonalization of Ill-Conditioned Systems, IEEE Trans. Autom. Contr., AC-20 (1975), 812-814.
31. M. Balas, Lecture Notes on Control of Large Mechanically Flexible Structures, (unpublished manuscript).
32. P. Kokotović, Subsystems, Time-Scales, and Multi-Modeling, IFAC Symp. on Large Scale Systems, Toulouse, France, 1980.

# Appendix I: Proof of Lemma 1

We want to apply the contraction mapping theorem to  $h$  to obtain  $L^*$ .

Consider any  $L$  in  $\Omega$  and note that

$$||L|| \leq ||L - H_{22}^{-1} H_{21}|| + ||H_{22}^{-1} H_{21}|| \leq 1 + M_5 \quad (I.1)$$

$$\text{Also, } ||h(L) - H_{22}^{-1} H_{21}|| = \epsilon ||H_{22}^{-1} L (H_{11} - H_{12} L)||$$

$$\leq \epsilon M_4 ||L|| (M_1 + M_2 ||L||)$$

$$\leq \epsilon M_4 (1 + M_5) (M_2 + M_6) \quad (I.2)$$

because  $M_6 = M_1 + M_2 M_5$  in (4.5). Therefore  $h(L)$  is in  $\Omega$  when  $0 < \epsilon \leq \epsilon_A$  where

$$\epsilon_A = [M_4 (1 + M_5) (M_2 + M_6)]^{-1} \quad (I.3)$$

because  $||h(L) - H_{22}^{-1} H_{21}|| \leq 1$ .

Consider  $L$  and  $L'$  in  $\Omega$  and, from (I.1),

$$||h(L) - h(L')|| = \epsilon ||H_{22}^{-1} [L(H_{11} - H_{12} L) - L'(H_{11} - H_{12} L')]]||$$

$$\leq \epsilon M_4 (||L(H_{11} - H_{12} L) - L(H_{11} - H_{12} L')|| +$$

$$||L(H_{11} - H_{12} L') - L'(H_{11} - H_{12} L')||)$$

$$\leq \epsilon M_4 (||L H_{12}|| + ||H_{11} - H_{12} L'||) ||L - L'||$$

$$\leq \epsilon M_4 (2(1 + M_5) M_2 + M_1) ||L - L'|| < ||L - L'|| \quad (I.4)$$

when  $0 < \epsilon < \epsilon_B$  where

$$\epsilon_B = [M_4 (M_1 + 2(1 + M_5) M_2)]^{-1} \quad (I.5)$$

But, from (I.3),

$$\begin{aligned}\epsilon_A &= [(1 + M_5) M_4 (M_1 + M_2 (1 + M_5))]^{-1} \\ &\leq [(1 + M_5) M_4 (M_1 + 2 M_2 (1 + M_5))]^{-1} \\ &= \epsilon_B / (1 + M_5).\end{aligned}$$

Therefore,  $\epsilon_A \leq (1 + M_5) \epsilon_A = \epsilon_B$ ; hence, we choose  $\epsilon_1 < \epsilon_A$  and, from (I.4),  $h$  is a contraction on  $\Omega$ . By the contraction mapping theorem ([29] Theo. II.1.1, p. 24),  $h$  has a unique fixed point  $L^*$  in  $\Omega$  for any  $0 < \epsilon < \epsilon_1$ , i.e.

$$L^* = h(L^*) = H_{22}^{-1} H_{21} + \epsilon H_{22}^{-1} L^* (H_{11} - H_{12} L^*) \quad (I.6)$$

and, from (I.1),

$$||L^*|| \leq 1 + M_5 \quad (I.7)$$

Note that  $H_{11} - H_{12} L^* = H_1 - \epsilon W_1$  and, by substitution of  $z_2$  into (4.1), obtain  $\dot{w}_1 = H_{11} w_1 + H_{12} (z_2 - L^* w_1) = (H_1 - \epsilon W_1) w_1 + H_{12} z_2$  and

$$\begin{aligned}\epsilon \dot{z}_2 &= \epsilon (\dot{w}_2 + L^* w_1) \\ &= H_{21} w_1 + H_{22} w_2 + \epsilon L^* ((H_1 - \epsilon W_1) w_1 + H_{12} z_2) \\ &= H_{22} [h(L^*) - L^*] w_1 + (H_{22} + \epsilon L^* H_{12}) z_2 \\ &= (H_{22} + \epsilon L^* H_{12}) z_2 \quad \text{by (I.6).}\end{aligned}$$

This is the result desired. #



## Appendix II: Proof of Lemma 2

Assume  $0 < \epsilon < \epsilon_0 < \epsilon_1$  and, from Lemma 1, obtain

$$\dot{w}_1 = (H_1 - \epsilon W_1) w_1 + H_{12} z_2 \quad (II.1)$$

$$\epsilon \dot{z}_2 = (H_{22} + \epsilon L^* H_{12}) z_2 \quad (II.2)$$

Choose  $\tau = t/\epsilon$  and  $\hat{z}_2(\tau) = z_2(t)$ . Therefore, from (II.2),

$$\frac{d\hat{z}_2}{d\tau} = \dot{z}_2 \frac{dt}{d\tau} = \epsilon \dot{z}_2 = (H_{22} + \epsilon L^* H_{12}) \hat{z}_2.$$

Hence, from  $H_{22} = A_R$  and (2.4) and [27], p. 215,

$$||\hat{z}_2(\tau)|| \leq K_R e^{-\hat{\delta}_2 \tau} ||\hat{z}_2(0)|| \quad (II.3)$$

where  $\hat{\delta}_2 = \delta_R - \epsilon_0 (1 + M_5) M_2 K_R$  because  $\epsilon ||L^* H_{12}|| < \epsilon_0 (1 + M_5) M_2$ .

Choose

$$\epsilon_0 < \delta_R [(1 + M_5) M_2 K_R]^{-1} \quad (II.4)$$

so that  $\hat{\delta}_2 > 0$ . From (II.3) and  $||z_2(t)|| = ||\hat{z}_2(\tau)||$ , obtain

$$\begin{aligned} ||z_2(t)|| &\leq K_R e^{-\frac{\hat{\delta}_2}{\epsilon} t} ||z_2(0)|| \\ &\leq K_R e^{-\frac{\hat{\delta}_2}{\epsilon_0} t} ||z_2(0)|| \end{aligned} \quad (II.5)$$

because  $\epsilon < \epsilon_0$ .

$$\text{Choose } \epsilon_0 < \delta_1 [(1 + M_5) M_2 M_4 (M_2 + M_6) K_1]^{-1} \quad (II.6)$$

where  $\delta_1$  and  $K_1$  are given in (4.4); assume  $\epsilon_0$  satisfies (II.4), as well. Let

$\hat{\delta}_1 = \delta_1 - \epsilon_0 K_1 M_2 M_4 (1 + M_5) (M_2 + M_6)$  which is positive by (II.6). Since

$||w_1|| \leq M_2 M_4 (1 + M_5) (M_2 + M_6)$ , we have, from (4.4), (II.1), (II.5), and

[28], p. 215,

$$\begin{aligned}
||w_1(t)|| &\leq K_1 e^{-\hat{\delta}_1 t} [||w_1(0)|| + M_2 \int_0^t e^{\hat{\delta}_1 s} ||z_2(s)|| ds] \\
&\leq K_1 e^{-\hat{\delta}_1 t} [||w_1(0)|| + M_2 K_R \int_0^t e^{\Delta s} ds ||z_2(0)||] \\
&\leq K_1 e^{-\hat{\delta}_1 t} [||w_1(0)|| + M_2 K_R \frac{e^{\Delta t} - 1}{\Delta}] \quad (II.7)
\end{aligned}$$

where  $\Delta = \hat{\delta}_1 - \frac{\hat{\delta}_2}{\epsilon_0}$ .

If  $\Delta > 0$ , then (II.7) implies  $||w_1(t)|| \leq K_1 e^{-\frac{\hat{\delta}_2}{\epsilon_0} t} (||w_1(0)|| + \frac{M_2 K_R}{\Delta} ||z_2(0)||)$ . If  $\Delta < 0$ , then (II.7) implies  $||w_1(t)|| \leq K_1 e^{-\hat{\delta}_1 t} (||w_1(0)|| + \frac{M_2 K_R}{|\Delta|} ||z_2(0)||)$ . In either case, if  $\delta_c = \min(\hat{\delta}_1, \frac{\hat{\delta}_2}{\epsilon_0})$ , then

$$||w_1(t)|| \leq K_1 e^{-\delta_c t} (||w_1(0)|| + \frac{M_2 K_R}{|\Delta|} ||z_2(0)||) \quad (II.8)$$

and, from (II.5) and (II.8) and  $K_1$  and  $K_R \geq 1$ ,

$$\begin{aligned}
||w_1(t)||^2 + ||z_2(t)||^2 &\leq (K_1 K_R e^{-\delta_c t})^2 (||z_2(0)||^2 + (||w_1(0)|| \\
&\quad + \frac{M_2}{|\Delta|} ||z_2(0)||)^2) \leq (K_1 K_R e^{-\delta_c t})^2 (1 + \alpha + \alpha^2) \\
&\quad (||w_1(0)||^2 + ||z_2(0)||^2)
\end{aligned}$$

where  $\alpha = \frac{M_2}{|\Delta|}$  and the easily verified inequality:

$$b^2 + (a + b)^2 \leq (1 + \alpha + \alpha^2) (a^2 + b^2)$$

is used with  $a = ||w_1(0)||$  and  $b = ||z_2(0)||$ . Therefore,

$$(||w_1(t)||^2 + ||z_2(t)||^2)^{1/2} \leq \tilde{K}_c e^{-\delta_c t} (||w_1(0)||^2 + ||z_2(0)||^2)^{1/2} \quad (II.9)$$

We must choose  $\epsilon_0$  to satisfy (II.4), (II.6) and  $\epsilon_0 < \epsilon_1 = \gamma \cdot$

$[M_4 (1 + M_5) (M_2 + M_6)]^{-1}$ , where  $0 < \gamma < 1$ . Note that, from (4.2),  $M_4 = K_R/\delta_R$ . Therefore choose  $\epsilon_0$  to satisfy:

$$\epsilon_0 < \frac{\delta_R}{K_R (1 + M_5) M_2} \cdot \min \left( 1, \frac{\delta_1}{K_1 (M_2 + M_6)} \cdot \frac{\gamma M_2}{M_2 + M_6} \right) \quad (\text{II.10})$$

and this will meet all the requirements. However, we can refine (II.10) further because  $0 < \gamma < 1$  and  $M_2[M_2 + M_6]^{-1} \leq 1$ ; consequently, (II.10) becomes:

$$\epsilon_0 < \frac{\delta_R}{K_R M_2 (1 + M_5) (M_2 + M_6)} \min \left( \frac{\delta_1}{K_1}, \gamma M_2 \right) \quad (\text{II.11})$$

which is the desired result. #

Appendix III: Proof of Theo. 1

Assume  $0 < \epsilon \leq \epsilon_0$ , where  $\epsilon_0$  given by (4.8). Consider that, from

$$z_2 = w_2 + L^* w_1,$$

$$\begin{bmatrix} w_1 \\ z_2 \end{bmatrix} = Q \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = Q w \quad (\text{III.1})$$

where

$$Q = \begin{bmatrix} I & 0 \\ L^* & I \end{bmatrix} \quad \text{and} \quad Q^{-1} = \begin{bmatrix} I & 0 \\ -L^* & I \end{bmatrix}$$

From Lemma 2, (II.9) and (III.1), obtain

$$||w(t)|| \leq ||Q^{-1}|| ||Q|| \tilde{K}_c e^{-\delta_c t} ||w(0)|| \quad (\text{III.2})$$

$$\begin{aligned} \text{But } ||Qw|| &= \sqrt{||w_1||^2 + ||w_2 + L^* w_1||^2} \\ &\leq (1 + ||L^*|| + ||L^*||^2)^{1/2} ||w||^2 \end{aligned} \quad (\text{III.3})$$

by use of the easily verified inequality:  $a^2 + (b + \alpha a)^2 \leq (1 + \alpha + \alpha^2) \cdot$

$(a^2 + b^2)$  with  $a = ||w_1||$ ,  $b = ||w_2||$ , and  $\alpha = ||L^*||$ . Therefore  $||Q|| \leq (1 + ||L^*|| + ||L^*||^2)^{1/2}$  and, similarly,  $||Q^{-1}|| \leq (1 + ||L^*|| + ||L^*||^2)^{1/2}$ .

Note that, from Lemma 1 (I.7),  $||L^*|| \leq 1 + M_5$ . When these inequalities are substituted into (III.2), it is clear that the desired result is obtained. #

# D. STABILITY OF DISTRIBUTED PARAMETER SYSTEMS WITH FINITE DIMENSIONAL COMPENSATORS VIA SINGULAR PERTURBATION METHODS

Mark J. Balas

Associate Professor  
Electrical, Computer, & Systems Engr. Dept.  
Rensselaer Polytechnic Institute  
Troy, NY 12181 USA

## ABSTRACT

Using a singular perturbation formulation of the linear time-invariant distributed parameter system, we develop a method to design finite-dimensional feedback compensators of any fixed order which will stabilize the infinite-dimensional distributed parameter system. The synthesis conditions are given entirely in terms of a finite-dimensional reduced-order model; the stability result depends on an infinite-dimensional version of the Klimushchev-Krasovskii lemma developed in [10].

## 1.0 INTRODUCTION

Many engineering systems exhibit a distributed parameter nature and must be described by partial differential equations. Examples of such distributed parameter systems (DPS) include heat diffusion & chemical processes, wave propagation, & mechanically flexible structures. Various aspects of the control of DPS have been considered in, for example, [1] - [5]; our experience in DPS has been shaped by applications in large aerospace structures [6].

The state spaces for DPS have infinite dimension; so, at best, reduced-order models must be used in controller synthesis. However, the closed-loop stability of the infinite dimensional DPS with a finite dimensional feedback controller becomes a fundamental issue. The synthesis of finite dimensional controllers for DPS and the analysis of their closed-loop stability by singular and regular perturbation techniques have been our main areas of emphasis [7]; this theory has been developed with flexible structures and other highly oscillatory DPS applications in mind.

Even in large-scale, lumped parameter systems, such as electric power distribution networks, it is necessary to perform model reduction and reduced-order controller synthesis and to analyze closed-loop stability. The use of asymptotic methods, especially singular perturbations, has been very successful in this regard (e.g. [8]). We have extended certain of these singular perturbations methods for DPS to provide estimates of stability in an infinite dimensional setting [9] - [10] and applied them to mechanically flexible structures [11]. In this paper we will use these singular perturbation results to analyze the general finite dimensional compensator for linear DPS.

We consider linear, time-invariant DPS with the form:

$$\begin{cases} \dot{v}_t = Av + Bf, & v(0) = v_0 \\ y = Cv \end{cases} \quad (1.1)$$

where the state  $v(t)$  is in a Hilbert space  $H$  (called the state space) for all  $t \geq 0$ ; the state space has inner product  $(\cdot, \cdot)$  and associated norm  $\|\cdot\|$ . The control vector  $f$  and the observation vector  $y$  have dimensions  $M$  &  $P$ , respectively, which denote the number of (independent) actuators and sensors in use; thus, rank  $B=M$  and rank  $C=P$ . The operator  $A$  is an unbounded differential operator with domain  $D(A)$  dense in  $H$  and  $A$  generates a  $C_0$  semigroup  $U(t)$  on  $H$ . Many practical DPS can be stated in this form [7].

## 2.0 REDUCED-ORDER MODELING OF DPS: A SINGULAR PERTURBATIONS FORMULATION

Since the state space  $H$  of the DPS in (1.1) is infinite dimensional, we must obtain a reduced-order model (ROM) upon which to base the finite dimensional controller design. In general, this is done by selecting a finite dimensional subspace.  $H_N$  (with  $\dim H_N = N < \infty$ ) contained in  $D(A)$ . This subspace  $H_N$  is the ROM subspace; its complement  $H_R$  is the Residual subspace, and together they decompose the state space:

$$H = H_N \oplus H_R \quad (2.1)$$

We define the (not necessarily orthogonal) projection operators  $P_N$  and  $P_R$  and let  $v_N = P_N v$  and  $v_R = P_R v$ . These decompose the state  $v = v_N + v_R$  and the DPS (1.1).

In this paper, we will make two basic assumptions on model reduction processes

- (1) the subspaces  $H_N, H_R$  are reducing subspaces (i.e. they are both  $A$ -invariant);
- (2) the subspaces  $H_N, H_R$  can be chosen to give a singular perturbations formulation with small parameter  $\epsilon > 0$  and an exponentially-stable residual subsystem.

These assumptions yield the decomposed DPS (1.1) in the following form:

$$\frac{\partial v_N}{\partial t} = A_N v_N + B_N f \quad (2.2a)$$

$$\epsilon \frac{\partial v_R}{\partial t} = A_R v_R + B_R f \quad (2.2b)$$

$$y = C_N v_N + C_R v_R \quad (2.2c)$$

where  $A_N = P_N A P_N$ ,  $B_N = P_N B$ ,  $C_N = C P_N$ , etc. and  $A_R$  generates the  $C_0$  semigroup  $U_R(t) = P_R U(t) P_R$  with exponential growth condition:

$$\|U_R(t)\| < K_R e^{-\sigma_R t}, \quad t \geq 0 \quad (2.3)$$

where  $K_R \geq 1$  and  $\sigma_R > 0$ . All operators except  $A_R$  are bounded. The terms  $B_R f$  &  $C_R v_R$  are called, respectively, control & observation spillover; they represent the interconnections through which the controller can affect the residual subsystem.

The ROM of the DPS, in this formation, is obtained by setting  $\epsilon=0$ :

$$\begin{cases} \frac{\partial v_N}{\partial t} = A_N v_N + B_N f \\ y = C_N v_N + D_N f \end{cases} \quad (2.4)$$

where  $D_N = -C_N A_N^{-1} B_N$  &  $A_N^{-1}$  is a bounded operator due to (2.3). The ROM is a finite dimensional system and the parameters  $(A_N, B_N, C_N, D_N)$  may be identified with matrices by choosing a basis in  $H_N$ ; this will be done wherever necessary. We shall assume, henceforth, that the ROM is controllable and observable (easily verifiable conditions exist for finite dimensional systems). The ROM is completely determined by the choice of the subspaces  $H_N$  &  $H_R$ .

### 3.0 FINITE DIMENSIONAL COMPENSATORS FOR DPS

The form of the finite dimensional feedback compensators used here will be the following:

$$\begin{cases} \dot{f} = L_{11}y + L_{12}z \\ \dot{z} = L_{21}y + L_{22}z \end{cases} \quad (3.1)$$

where the compensator state  $z$  has  $\dim. z = \alpha \leq N < \infty$  and  $L_{11}, L_{12}, L_{21}, L_{22}$  are matrices of appropriate sizes. We say the compensator is output feedback when  $\alpha=0$ . The order  $\alpha$  of the compensator is assumed to be fixed at some acceptable value which reflects the available capacity of the on-line computer being used to implement (3.1). In [10], the compensator order was  $\alpha = N$ .

The compensator design is synthesized as though the ROM (2.4) were the full DPS (1.1), i.e. as though  $\epsilon=0$ . Let

$$F = \bar{A}_N + \bar{B}_N \bar{L} \bar{C}_N \quad (3.2)$$

where

$$\bar{A}_N = \begin{bmatrix} A_N & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_N = \begin{bmatrix} B_N & 0 \\ 0 & I_\alpha \end{bmatrix}, \quad \bar{C}_N = \begin{bmatrix} C_N & 0 \\ 0 & I_\alpha \end{bmatrix}, \quad \& \quad \bar{L} = \begin{bmatrix} \bar{L}_{11} & \bar{L}_{12} \\ \bar{L}_{21} & \bar{L}_{22} \end{bmatrix}$$

with

$$\bar{L}_{11} = (I_M - L_{11} D_N)^{-1} L_{11}$$

$$\bar{L}_{12} = (I_M - L_{11} D_N)^{-1} L_{12}$$

$$\bar{L}_{21} = L_{21} (I_P + D_N \bar{L}_{11})$$

$$\& \quad \bar{L}_{22} = L_{22} + L_{21} D_N \bar{L}_{12}$$

Let the composite state  $q_N = \begin{bmatrix} v_N \\ z \end{bmatrix}$  in  $H_N \times R^\alpha$ ; from (2.4) & (3.1), this satisfies (for  $\epsilon=0$ ):

$$\frac{\partial q_N}{\partial t} = F q_N \quad (3.3)$$

The following theorem gives the conditions under which a stable design can be synthesized:

Theorem 3.1: If the ROM  $(A_N, B_N, C_N, D_N)$  is controllable and observable and

$$M + P + \alpha \geq N + 1 \quad (3.4)$$

then  $\bar{L}$  may be chosen so that  $F$  given by (3.2) has any desired pole locations in

the complex plane.

The proof follows easily from results in [12] or [13] and depends on the finite dimensionality of the closed-loop system when  $\varepsilon=0$ . Since the gains  $\bar{L}$  can be synthesized, the compensator gains  $L_{11}, L_{12}, L_{21}, L_{22}$  can be found; however, we must assume that  $\|D_N\|$  is sufficiently small to make both  $(I_M - L_{11} D_N)^{-1}$  &  $(I_P + D_N \bar{L}_{11})^{-1}$  exist.

#### 4.0 CLOSED-LOOP STABILITY FOR THE DPS WITH A FINITE-DIMENSIONAL COMPENSATOR

Let  $\varepsilon$  be such that  $0 < \varepsilon < \varepsilon_0$ , and consider the closed-loop system consisting of the infinite-dimensional DPS (2.2) and the finite-dimensional compensator (3.1). Let the closed-loop state  $w$  in  $\bar{H} = H_N \times R^q \times H_R$  be given by

$$w = \begin{bmatrix} w_N \\ w_R \end{bmatrix}$$

where  $w_N = \begin{bmatrix} v_N \\ z \end{bmatrix}$  is in  $H_N \times R^q$  &  $w_R = v_R$  in  $H_R$ . The norm on  $\bar{H}$  is defined by

$$\|w\| = (\|v_N\|^2 + \|z\|^2 + \|v_R\|^2)^{1/2} \quad (4.1)$$

From (2.2) & (3.1), the closed-loop state satisfies:

$$\frac{\partial w_N}{\partial \tau} = H_{11} w_N + H_{12} w_R \quad (4.2a)$$

$$\varepsilon \frac{\partial w_R}{\partial \tau} = H_{21} w_N + H_{22} w_R \quad (4.2b)$$

where

$$H_{11} = \begin{bmatrix} A_N + B_N L_{11} C_N & B_N L_{12} \\ L_{21} C_N & L_{22} \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} B_N L_{11} C_R \\ L_{21} C_R \end{bmatrix}$$

$$H_{21} = [B_R L_{11} C_N \quad B_R L_{12}]$$

$$H_{22} = A_R + B_R L_{11} C_R$$

But, (4.2) is in the form of the infinite dimensional Klimushchev-Krasovskii lemma whose proof appears in the revised version of [10]; estimates on the upper bound  $\varepsilon_0$  are presented there, as well. From the infinite-dimensional K-K lemma, we obtain the following closed-loop stability result:

Theorem 4.1: If  $H_{22}$  generates an exponentially stable  $C_0$  semigroup and the finite rank operator

$$\bar{H} = H_{11} - H_{12} H_{22}^{-1} H_{21} \quad (4.3)$$

has all its poles in the open left half-plane, then there exists positive  $\varepsilon_0$  (whose



value can be estimated from the model data) such that, for all  $0 < \epsilon \leq \epsilon_0$ , (4.2) is stable.

This theorem says that under certain conditions the finite dimensional controller based on the ROM ( $\epsilon=0$ ) will continue to stabilize the infinite dimensional DPS when  $\epsilon$  is "small" but non-zero. The "smallness" required of  $\epsilon$  can be estimated from bounds on the control and observation spillover terms, the constants  $K_R, \sigma_R$  in (2.3), and the ROM data.

The following lemma greatly simplifies Theo. 4.1:

Lemma 4.2:  $\bar{H} = F = \bar{A}_N + \bar{B}_N \bar{L}_N \bar{C}_N$  (4.4)

The proof appears in the Appendix. Consequently, Theo. 4.1 may be restated as our main result:

Theorem 4.3: If the stable compensator synthesis conditions of Theo. 3.1 are satisfied for the ROM (2.4) and if the control and observation spillover coefficients  $\beta = \|B_R\|$  &  $\Gamma = \|C_R\|$  are sufficiently small that

$$\frac{C_R}{K_R} > \beta \Gamma \|L_{11}\| \quad (4.5)$$

where  $K_R$  &  $\sigma_R$  appear in (2.3), then there exists positive  $\epsilon_0$  such that for all  $0 < \epsilon \leq \epsilon_0$  the same finite-dimensional compensator (3.1) stabilizes the infinite-dimensional DPS (1.1).

This follows from Lemma 4.2 and use of the semigroup-generator perturbation result ([14], Theo. 10.9) applied to

$$H_{22} = A_R + B_R L_{11} C_R$$

because  $B_R L_{11} C_R$  is a bounded operator.

## 5.0 CONCLUSIONS

Our main result (Theo. 4.3) gives conditions under which a general finite-dimensional compensator, based on a singular perturbations reduced-order model, will stabilize an infinite dimensional (linear) distributed parameter system. This is an extremely useful result since all practical feedback compensators must be finite dimensional in order to be implementable with on-line digital computers. The result is valid for large-scale lumped-parameter systems, as well.

The most difficult assumption to satisfy is the choice of reduced-order subspace  $E_N$  to achieve a singularly perturbed formulation (2.2) of the distributed parameter system (1.1); the other assumptions are reasonably easy to satisfy. Modal methods have worked for mechanically flexible structures, e.g. [11]. A general discussion of this modeling difficulty is given in [15]; it is a fundamental problem in large-scale or distributed parameter systems and should not be overlooked.

## ACKNOWLEDGEMENT

This work was supported by NASA Contract No. NAS-9-16053.

1. Lions, J.L. (1972). Some Aspects of the Optimal Control of DPS, SIAM, Philadelphia, PA.
2. Ray, W.H., and Lainiotis, D., eds., (1978). DPS: Identification, Estimation, & Control, M. Dekker, NY.
3. Russell, D., (1978). Controllability & Stabilizability Theory for Linear PDE: Recent Progress & Open Questions, SIAM Review, 20, 371-388.
4. Curtain, R., and Pritchard, A., (1978). Infinite Dimensional Systems Theory, Springer, NY.
5. Aziz, A., et.al. eds., (1977). Control Theory of Systems Governed by PDE, Academic Press, NY.
6. Balas, M., Some Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams, IEEE Trans. Autom. Control (to appear).
7. Balas, M., (1980). Toward A (More) Practical Theory of Distributed Parameter System Control, Proc. Int'l. Conf. on Computers & Circuits, Port Chester, NY.
8. Kokotović, P., O'Malley, R., and Sannuti, P., (1976). Singular Perturbations & Order Reduction in Control Theory: An Overview, Automatica, 12, 123-132.
9. Balas, M., Singular Perturbations & Model Reduction Methods For Large Scale & Distributed Parameter Systems, submitted to J. Franklin Inst.
10. Balas, M., (1980). Reduced-Order Feedback Control of DPS Via Singular Perturbations Methods, Princeton Conf. on Information Sciences & Systems, Princeton, NJ; revised version submitted to SIAM J. Contr. & Opt.
11. Balas, M., (1980). Closed-Loop Stability of Large Space Structures Via Singular & Regular Perturbation Techniques: New Results, IEEE Control & Decision Conf., Albuquerque, N.M.
12. Kimura, H., (1975). Pole Assignment by Gain Output Feedback, IEEE Trans. Autom. Control, AC-20, 509-516.
13. Davison, E., and Wang, S. On Pole Assignment in Linear Multivariable Systems Using Output Feedback, Ibid, 516-518.
14. Curtain, R., and Pritchard, A., (1977). Functional Analysis in Modern Applied Mathematics, Academic Press, NY.
15. Kokotović, P., (1980). Subsystems, Time-Scales, & Multi-Modeling, IFAC Symp. on Large Scale Systems, Toulouse, France.

#### APPENDIX: PROOF OF LEMMA 4.2

$$\bar{H} = H_{11} - H_{12} H_{22}^{-1} H_{21}$$

$$= \begin{bmatrix} A_N + B_N Z L_{11} C_N & B_N Z L_{12} \\ L_{21} (I_P - C_R H_{22}^{-1} B_R L_{11}) C_N & L_{22} - L_{21} C_R H_{22}^{-1} B_R L_{12} \end{bmatrix}$$

where  $Z = I_M - L_{11} C_R H_{22}^{-1} B_R$

&  $H_{22} = A_R + B_R L_{11} C_R$

$$F = \bar{A}_N + \bar{B}_N \bar{L} \bar{C}_N$$

$$= \begin{bmatrix} A_N + B_N \bar{L}_{11} C_N & B_N \bar{L}_{12} \\ \bar{L}_{21} C_N & \bar{L}_{22} \end{bmatrix}$$

where

$$\begin{aligned}\bar{L}_{11} &= (I_M - L_{11}D_N)^{-1} L_{11} \\ \bar{L}_{12} &= (I_M - L_{11}D_N)^{-1} L_{12} \\ \bar{L}_{21} &= L_{21}(I_P + D_N\bar{L}_{11}) \\ \bar{L}_{22} &= L_{22} + L_{21}D_N\bar{L}_{12} \\ \& \quad D_N &= -C_R A_R^{-1} B_R\end{aligned}$$

Therefore,  $\bar{H} = F$  if and only if

(a)  $Z = (I_M - L_{11}D_N)^{-1}$  and  
 (b)  $C_R H_{22}^{-1} B_R = -D_N (I_M - L_{11}D_N)^{-1}$

However, (a)  $\rightarrow$  (b):

$$\begin{aligned}-D_N (I_M - L_{11}D_N)^{-1} &= -D_N Z = -D_N + D_N L_{11} C_R H_{22}^{-1} B_R \\ &= C_R A_R^{-1} (I_R - B_R L_{11} C_R H_{22}^{-1}) B_R \\ &= C_R A_R^{-1} (H_{22} H_{22}^{-1} - B_R L_{11} C_R H_{22}^{-1}) B_R \\ &= C_R A_R^{-1} (H_{22} - B_R L_{11} C_R) H_{22}^{-1} B_R \\ &= C_R A_R^{-1} (A_R) H_{22}^{-1} B_R = C_R H_{22}^{-1} B_R\end{aligned}$$

Thus it remains to prove (a) which we recast as:

(a')  $(I_M + QW)^{-1} = I_M - Q(WQ + I_R)^{-1} W$

where  $Q = L_{11}C_R$  and  $W = A_R^{-1} B_R$  are both bounded (finite rank) operators.

Since  $(WQ + I_R)^{-1} = (I_R + A_R^{-1} B_R L_{11} C_R)^{-1} = H_{22}^{-1} A_R$  and  $H_{22}^{-1}$  is assumed to exist for small enough spillover bounds, therefore

$$Z = I_M - Q(WQ + I_R)^{-1} W$$

where the required inverse exists. Thus, to prove (a) we need only prove (a'); however, this follows directly:

$$\begin{aligned}(I_M + QW) (I_M - Q(WQ + I_R)^{-1} W) \\ &= I_M + QW - Q(WQ + I_R)^{-1} W \\ &\quad - Q[WQ(WQ + I_R)^{-1} W] \\ &= I_M + QW - Q(WQ + I_R)^{-1} W \\ &\quad - Q[I_R - (WQ + I_R)^{-1}] W \\ &= I_M + QW - Q(WQ + I_R)^{-1} W \\ &\quad - QW + Q(WQ + I_R)^{-1} W = I_M \\ \therefore (I_M + QW)^{-1} &= I_M - Q(WQ + I_R)^{-1} W\end{aligned}$$

and (a') is proved.

# E. CLOSED-LOOP STABILITY OF LARGE SPACE STRUCTURES VIA SINGULAR AND REGULAR PERTURBATION TECHNIQUES: NEW RESULTS

Mark J. Balas  
Associate Professor  
Electrical and Systems Engineering Department  
Rensselaer Polytechnic Institute  
Troy, New York 12181  
(518) 270-6485

## Abstract

Large Space Structures are treated as a special class of highly oscillatory distributed parameter systems. Practical controllers will need to be finite dimensional; however, this calls the closed-loop stability into question due to spillover interactions. Recent results, obtained using regular and singular perturbations theory, give stability bounds for closed-loop control of this type of distributed parameter system; this paper surveys these results and their implications for large space structures.

## 1.0 Introduction

Large aerospace structures, such as satellites and spacecraft, are mechanically flexible structures which will usually require active (feedback) control systems to suppress vibrations and satisfy stringent requirements for pointing accuracy, orientation, shape, and station-keeping. The problems and current directions in large aerospace structure control have been surveyed, for example, in [1]-[3].

Such structures are highly oscillatory, distributed parameter systems (DPS). The viewpoint we have been pushing for DPS in general is a practical control theory, i.e. one that includes such practical constraints as a finite number of (localized) control devices and finite-dimensional (implementable) controllers; this has been discussed in [4]. Other helpful references on aspects of DPS theory are [5]-[8]. In this paper, large aerospace structure control will be viewed in the framework established in [4], and recent stability results using regular and singular perturbations techniques [9]-[10] will be examined in the large space structure context.

## 1.1 Large Aerospace Structure Description

Large aerospace structures may often be modeled by the following DPS description:

$$u_{tt}(x,t) - D_0 u_t(x,t) + A_0 u(x,t) = F_0(x,t) \quad (1.1)$$

where  $F_0(x,t)$  is the displacement of the structure  $\Omega \subset \mathbb{R}^n$  off its equilibrium position in space. The damping and stiffness differential operators  $D_0$  and  $A_0$  are defined on a domain  $D(A_0)$  which is a subspace of the Hilbert Space  $H_0 = L^2(\Omega)$  with inner product  $(\cdot, \cdot)_{H_0}$ ;  $D(A_0)$  contains all smooth functions in  $H_0$  which satisfy the structural boundary conditions. Furthermore, the operators satisfy:

$$A_0 \phi_k = -\omega_k^2 \phi_k \quad (1.2a)$$

$$D_0 \phi_k = 2\zeta_k \omega_k \phi_k \quad (1.2b)$$

where  $\phi_k(x)$  are the mode shapes,  $\omega_k$  the mode frequencies, and  $\zeta_k$  the modal damping (which is generally thought to be small for large aerospace structures).

Although the force distribution  $F_0(x,t)$  in (1.1) is the sum of disturbance and control forces, we will

ignore these external disturbances to simplify this discussion and consider the control forces as generated by  $M$  independent actuators:

$$F_0(x,t) = B_0 f = \sum_{j=1}^M b_j(x) f_j(t) \quad (1.3)$$

Observation is done with  $P$  independent sensors whose output vector is given by

$$y(t) = C_0 u(x,t) + E_0 u_t(x,t) \quad (1.4)$$

## 1.2 State Space Description

The DPS (1.1), and (1.3)-(1.4) can be rewritten in many state space formulations (not all of them are equivalent); however, the most natural one is the state  $v(x,t) = [u(x,t), u_t(x,t)]$  which satisfies:

$$\begin{cases} v_t = Av + Bf \\ y = Cv \end{cases} \quad (1.5)$$

with the state space being the Hilbert space  $H = D(A_0^{1/2}) \times H_0$  with the energy norm

$$\|v\|^2 = \|u_t\|_{H_0}^2 + \|A_0^{1/2} u\|_{H_0}^2 \quad (1.6)$$

and the operators  $(A, B, C)$  determined from (1.1) and (1.3)-(1.4). The input and output operators  $B$  and  $C$  have finite rank, and the system operator  $A$  has domain  $D(A) = D(A_0^{1/2}) \times D(A_0)$  and generates the  $C_0$  semigroup  $U(t)$ .

The damping present in the structure gives the system a small amount of overall dissipation. It is the job of the finite-dimensional feedback controller to redistribute this dissipation to critical structural modes to improve their performance without creating instabilities in any residual modes.

## 2.0 Model-Reduction and Controller Synthesis

The synthesis of a finite-dimensional controller is based on a reduced-order (finite-dimensional) model (ROM) of the DPS. In general this is obtained by decomposing the original state space  $H$  into the ROM subspace  $H_N$  and the residual subspace  $H_R$ :

$$H = H_N \oplus H_R \quad (2.1)$$

where  $\dim H_N < \infty$ . Then projections  $v_N = P_N v$  and  $v_R = P_R v$  are defined for these subspaces, and the DPS (1.5) is decomposed:

$$(v_N)_t = A_N v_N + A_{NR} v_R + B_N f \quad (2.2a)$$

$$e(v_R)_t = A_{RN} v_N + A_R v_R + B_R f \quad (2.2b)$$

$$y = C_N v_N + C_R v_R \quad (2.2c)$$

ORIGINAL PAGE IS  
OF POOR QUALITY

where  $A_N = P_N A P_N$ ,  $A_{NR} = P_N A P_R$ ,  $B_N = P_N B$ , etc. The terms  $A_{NR} v_R$  and  $A_{RN} v_N$  are called model error, and  $B_R f$ ,  $C_R v_R$  are called control and observation spillover, respectively. The parameter  $\epsilon > 0$  appears in some cases for appropriate choices of the subspaces  $H_N$  and  $H_R$ ; when it appears and is small, we are dealing with a singular perturbations formulation of the DPS. In many cases  $\epsilon = 1$  for the choice of subspaces and this is a regular perturbations formulation. Both of these formulations will be dealt with here. A basic, and reasonable, assumption made in both formulations is that  $A_R$  generates a stable  $C_0$  semigroup.

## 2.1 Regular Perturbations ROM and Controller Synthesis

When  $\epsilon = 1$ , the regular perturbations ROM is obtained from (2.2) by ignoring the residual state  $v_R$ :

$$(\dot{v}_N)_t = A_N v_N + B_N f \quad (2.3a)$$

$$y = C_N v_N \quad (2.3b)$$

This is a finite-dimensional subsystem (since  $\dim H_N < \infty$ ); consequently, the following finite-dimensional controller may be synthesized:

$$f = G_N \hat{v}_N \quad (2.4a)$$

$$(\dot{\hat{v}}_N)_t = A_N \hat{v}_N + B_N f + K_N (y - \hat{y}) \quad (2.4b)$$

$$\hat{y} = C_N \hat{v}_N \quad (2.4c)$$

The controller gains  $G_N$  and  $K_N$  are designed to stabilize and improve the performance of the ROM (2.3) in closed-loop with the controller (2.4). Of course the closed loop stability of the actual DPS (1.5) with the controller (2.4) will be a principal question; regular perturbation theory is used to answer this question in [9].

For large space structures, the most popular model reduction technique has been the modal one, i.e.

$$v_N = P_N v = \begin{bmatrix} P_N^0 u \\ P_N^0 u_c \end{bmatrix} \quad (2.5a)$$

$$v_R = P_R v = \begin{bmatrix} P_R^0 u \\ P_R^0 u_c \end{bmatrix} \quad (2.5b)$$

where  $P_N^0 u = \sum_{k=1}^N u_k \phi_k$ ,  $P_R^0 u = \sum_{k=N+1}^{\infty} u_k \phi_k$ , and  $u_k$  satisfies (by using (1.2) in (1.1)):

$$\ddot{u}_k + 2\zeta_k \dot{u}_k - \omega_k^2 u_k = F_k = (F_0, \phi_k) \quad (2.6)$$

In this formulation, the projections  $P_N$  and  $P_R$  are orthogonal in  $H$  and they "reduce"  $A$ , i.e. the model error terms  $A_{NR} = 0$  and  $A_{RN} = 0$ ; of course, this depends on knowing the true structure modes  $\phi_k$  rather than approximations of them. So the regular perturbations ROM (2.3) of the structure has parameters:

$$A_N = \begin{bmatrix} 0 & I_N \\ -A_N^0 & -2\zeta_N^0 \end{bmatrix}, \quad B_N = \begin{bmatrix} 0 \\ B_N^0 \end{bmatrix}, \quad C_N = \begin{bmatrix} C_N^0 & E_N^0 \end{bmatrix} \quad (2.7)$$

with  $A_N^0 = \text{diag} [\omega_1^2, \dots, \omega_N^2]$ ,  $D_N^0 = 2 \text{diag} [\zeta_1 \omega_1, \dots, \zeta_N \omega_N]$ , and  $B_N^0$ ,  $C_N^0$  have terms involving the inner products in  $H$  of actuator or sensor influence functions with mode shapes  $\phi_1, \dots, \phi_N$ . Corresponding parameters appear for the residual subsystem ( $A_R$ ,  $B_R$ ,  $C_R$ ) in (2.2).

The modal controller associated with this modal ROM can be obtained directly using the parameters (2.7) in (2.4). Modal controllability and observability conditions are available and easily verified, e.g. [3]; consequently, the controller synthesis may be carried out by standard finite-dimensional techniques. Note that any subset of critical modes may be used in this approach.

## 2.2 Singular Perturbations ROM and Controller Synthesis

The singular perturbations formulation of (2.2) occurs when  $\epsilon > 0$  and small. The singular perturbations ROM is obtained from (2.2) by setting  $\epsilon = 0$  and is given by

$$(\dot{v}_N)_t = \bar{A}_N v_N + \bar{B}_N f \quad (2.8a)$$

$$y = \bar{C}_N v_N + \bar{D} f \quad (2.8b)$$

where  $\bar{A}_N = A_N - A_{NR} A_R^{-1} A_{RN}$ ,  $\bar{B}_N = B_N - A_{NR} A_R^{-1} B_R$ ,  $\bar{C}_N = C_N - C_R A_R^{-1} A_{RN}$ , and  $\bar{D} = -C_R A_R^{-1} B_R$ . This is clearly different from the ROM obtained in (2.3). However, it is a finite-dimensional ROM and the controller associated with it is given by:

$$f = \bar{G}_N \hat{v}_N \quad (2.9a)$$

$$(\dot{\hat{v}}_N)_t = \bar{A}_N \hat{v}_N + \bar{B}_N f + \bar{K}_N (y - \hat{y}) \quad (2.9b)$$

$$\hat{y} = \bar{C}_N \hat{v}_N + \bar{D} f \quad (2.9c)$$

This controller differs from that in (2.4); however, the gains  $\bar{G}_N$  and  $\bar{K}_N$  are designed by standard finite-dimensional techniques to stabilize and improve the performance of the ROM (2.8) in closed-loop with the controller (2.9). Again, the closed-loop stability of the reduced-order controller (2.9) with the actual DPS when  $\epsilon$  is nonzero is in question. To attempt to answer this fundamental question a singular perturbations stability theory was developed in [10] for DPS of the form (2.2) with finite-dimensional controllers (2.9); this theory gives estimates of the tolerable size of  $\epsilon$ . The corresponding singular perturbations theory for finite dimensional systems is surveyed in [11]-[12].

For large space structures, as in most applications, there is no preferred singular perturbations formulation. The general approach has been applied to space structures in [13]. However, as pointed out in [14], the hardest task in the application of the singular perturbations approach is casting the problem in an appropriate singular formulation (2.2) where the small parameter  $\epsilon$  will have a connection with physical parameters and it will be possible to verify whether  $\epsilon$  is sufficiently small or not; this is true for applications of the approach to DPS, as well as lumped parameter systems.

Here we will mention two directions - time scaling and frequency scaling - which lead to singularly perturbed formulations of the modal model discussed for structures at the end of Sec. 2.1.

In the time-scaling approach (see [12] for a general

discussion), we assume that the structure modes of (2.6) separate into vibrations on two time scales - a slow one and a fast one - whose ratio is  $\epsilon$ , i.e. (2.6) becomes:

$$\ddot{u}_k + 2\epsilon\omega_k\dot{u}_k + \omega_k^2 u_k = F_k, \quad 1 \leq k \leq N \quad (2.10a)$$

$$\epsilon^2 \ddot{u}_k + \epsilon 2\epsilon\omega_k\dot{u}_k + \omega_k^2 u_k = F_k, \quad k \geq N+1 \quad (2.10b)$$

Some relabeling of modes might be required to obtain (2.10). Our associated singular perturbations formulation is obtained by taking  $v_N$  as in (2.5a) and, in place of  $v_R$ , using:

$$\bar{v}_R = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} v_R \quad (2.11)$$

where  $v_R$  is as in (2.5b). This yields a singular formulation (2.2) of the modal model of the structure with  $\epsilon$  being the ratio of the time-scales; the modal parameters are the same as those given in (2.7) with the exception that  $\bar{C}_R = [C_R^0 \quad \epsilon C_R^1]$  replaces  $C_R$  in the residual subsystem. The ROM (2.8) is obtained by setting  $\epsilon = 0$ , and the corresponding controller (2.9) can be synthesized; this controller will differ from the original modal controller (2.4).

In the frequency scaling approach (see [15] for a general discussion), we assume that  $\epsilon = 1/\omega_{N+1}$  for  $N$  sufficiently large, i.e. the residual structural modes have sufficient frequency separation from the ones contained in the ROM. In this case the associated singular perturbations formulation (2.2) is obtained by taking  $v_N$  as in (2.5a) and, in place of  $v_R$ , using:

$$\bar{v}_R = \begin{bmatrix} 1/\epsilon^2 & 0 \\ 0 & 1/\epsilon \end{bmatrix} v_R \quad (2.12)$$

where  $v_R$  is as in (2.5b); the modal parameters are the same as those in (2.7) with the exceptions:

$$\bar{A}_R = \begin{bmatrix} 0 & 1 \\ -\bar{\omega}_R^2 & -\bar{\omega}_R^2 \end{bmatrix} \text{ and } \bar{C}_R = [c^2 C_R^0 \quad c C_R^1] \text{ replace}$$

$A_R$  and  $C_R$ , where  $\bar{\omega}_R^2 = \text{diag} [\omega_{N+1}^2, \dots]$ .

$\bar{\omega}_R^2 = 2 \text{diag} [\omega_{N+1}, \omega_{N+1}, \dots]$ , and  $\bar{\omega}_k = \omega_k/\omega_{N+1}$  for  $k > N+1$ . Something interesting happens here when we set  $\epsilon = 0$  to obtain the ROM (2.8) and corresponding controller (2.9): since  $\bar{C}_R$  becomes zero when  $\epsilon = 0$  (and  $\bar{A}_R = 0$ ,  $\bar{\omega}_R = 0$  because of the use of structure modes), the singular perturbations ROM (2.8) is exactly the same as the regular perturbation one (2.3), i.e.

$\bar{A}_N = A_N$ ,  $\bar{B}_N = B_N$ ,  $\bar{C}_N = C_N$ , and  $\bar{D} = 0$ . Consequently, for the frequency scaling approach the controller (2.9) is exactly the same as the original modal controller, when sufficiently many modes are controlled. However, the singular perturbations stability results of [10] would seem to yield different conditions on the size of  $\epsilon = 1/\omega_{N+1}$  than the regular perturbation results of [9].

Other singular formulations of the problem may yield different results.

### 3.0 Conclusions

The stability of reduced-order finite dimensional controllers in closed-loop with distributed parameter systems (DPS) models of large aerospace structures has been considered here. General model reduction and

controller synthesis techniques for DPS have appeared in [9]-[10] with corresponding stability results obtained from both singular and regular perturbations formulations of the problem. These have been surveyed here by considering their implications for modal control of large aerospace structures. Both singular and regular perturbations formulations yield useful results, but the singular formulation seems extremely versatile due to the variety of singular formulations available in large space structure applications; two of these singular formulations - time and frequency scaling - were discussed here.

### References

1. S. Seltzer (editor), "Special Issue on Large Space Structures," J. Astro. Sci., 27 (1979).
2. R. Gran, "A Survey of the Large Structures Control Problem," Proc. of IEEE CDC, Ft. Lauderdale, Fla., 1979.
3. M. Balas, "Some Trends in Large Space Structure Control Theory: Fondest Hopes, Wildest Dreams," to appear in IEEE Trans. Autom. Contr.
4. M. Balas, "Distributed Parameter Feedback Control with Finite Dimensional Controllers: Recent Results," Proc. IEEE Internat'l Conf. on Circuits and Computers, Port Chester, NY, 1980.
5. R. Curtain and A. Pritchard, Infinite Dimensional Linear Systems Theory, Springer-Verlag, NY, 1978.
6. J. Walker, Dynamical Systems and Evolution Equations, Theory and Applications, Plenum Press, NY, 1980.
7. A. Balakrishnan, Applied Functional Analysis, Springer-Verlag, NY, 1978.
8. D. Russell, "Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions," SIAM Rev., 20 (1978), 371-388.
9. M. Balas, "Feedback Control of Dissipative Hyperbolic Distributed Parameter Systems," Proc. IEEE CDC, Ft. Lauderdale, Fla., 1979.
10. M. Balas, "Reduced-Order Feedback Control of Distributed Parameter Systems Via Singular Perturbations Methods," Conf. on Info. Sci. and Sys., Princeton Univ., NJ, 1980.
11. P. Kokotović, R. O'Malley, and P. Sannuti, "Singular Perturbations and Order Reduction in Control Theory: An Overview," Automatica, 12 (1976), 123-132.
12. P. Kokotović, J. Allemong, J. Winkelman, and J. Chow, "Singular Perturbation and Iterative Separation of Time Scales," Automatica, 16 (1980), 23-33.
13. J. Sezak, P. Likins, and T. Coradetti, "Flexible Spacecraft Control by Model Error Sensitivity Suppression," in [1].
14. P. Kokotović, Subsystems, Time Scales, and Multimodeling," 2nd IFAC Symp. on Large Scale Systems: Theory and Applications, Toulouse, France, 1980.
15. J. Chow, J. Allemong, and P. Kokotović, "Singular Perturbation Analysis of Systems with Sustained High Frequency Oscillations," Automatica, 14 (1978), 271-279.

## F. TOWARD ADAPTIVE CONTROL OF LARGE STRUCTURES IN SPACE

Mark J. Balas

Electrical & Systems Engineering Department  
Rensselaer Polytechnic Institute  
Troy, New York

C. Richard Johnson, Jr.<sup>1</sup>

Department of Electrical Engineering  
Virginia Polytechnic Institute & State University  
Blacksburg, Virginia

On-line adaptive control is essential for Large Space Structures (LSS) where the modal parameters are poorly known, due to modeling error, or changing, due to variable configurations. It is especially important that such adaptive controllers produce stabilizing controls during adaptation due to the small damping present in LSS. However, any such controller must be based on a reduced-order model of the LSS. The spillover from the unmodelled residuals, as well as the modeling error, can deteriorate the performance of the adaptive controller and, if uncompensated, this spillover can defeat the whole purpose of the adaptive control.

This paper investigates adaptive control for LSS using direct and indirect schemes and points out the mechanisms whereby observation spillover can upset the stability of the controller. The framework for nonadaptive control of LSS is reviewed and many of the generic problems of adaptive LSS control are pointed out within this framework. These generic problems must be overcome for successful operation of adaptive LSS control.

### 1.0 INTRODUCTION

This paper deals with the basic problems inherent in adaptive control of large space structures (LSS), such as satellites and spacecraft, where the structural parameters are poorly known or slowly time-varying.

With the advent of the Space Shuttle Transportation System, it has become possible to conceive very large spacecraft and satellites which

---

<sup>1</sup>Support by NASA Grant No. NAG-1-7.

could be carried into space and deployed, assembled, or manufactured there. Such LSS would serve a variety of civilian and military needs [1], [2], including electrical energy generation from the solar power satellite - a structure nearly the size of Manhattan Island - to be constructed in space and operated in earth geosynchronous orbit [3]. The control technology needs for such LSS have been discussed in a variety of articles, e.g., [4], [5], and the developing LSS control theory and technology has been surveyed, for example, in [6]-[10].

The size of these structures, their low rigidity, and the small damping available in lightweight construction materials combine to make LSS extremely mechanically flexible. In theory, LSS are distributed parameter systems whose dimension is infinite; however, in practice, their dynamics are usually modeled by large scale systems based on approximate elastic mode data. Active control schemes for LSS are often required to meet stringent requirements for their shape, orientation, alignment, and pointing accuracy. Such active control is limited by the capacity of the on-board control computer, the modeling inaccuracy in current finite element computer codes for analyzing structural dynamics, and available control devices (actuators and sensors); therefore, the controller must be based on some reduced-order model (ROM) of the LSS.

Fundamental problems of LSS control include:

- (1) selection of appropriate modes to control for desired system performance;
- (2) development of ROM for analysis and controller design;
- (3) computation of system model and control parameters;
- (4) controller design with multiple distributed actuators and sensors;
- (5) the number and location of sensors and actuators for efficient control;
- (6) the effect of, and compensation for, residual (unmodelled) modes and modeling error on the closed-loop system performance;
- (7) adaptive and self-tuning controllers for LSS with poorly known or changing parameters and configurations.

Item (7) is the basic topic of this paper but it must be considered in the context of the other items with which it is completely intertwined.

The need for adaptive control in LSS arises because of ignorance of the system and changing control regimes. The former is due to



(a) ignorance of the system structure and order, and (b) ignorance of the system parameters; the latter occurs because of changing configuration of the LSS. Changes in configuration may be due to construction in-space, thermal distortion, or reorientation of subsystems, e.g., rotating solar panels or sunshields; these changes usually produce slowly time-varying parameters. Ignorance of the LSS system structure and order is due to the fundamental problem of modeling a distributed parameter system, e.g., faulty physics, reduced-order models, and ignored nonlinearities; this means that the order of the ROM is lower than that of the actual LSS. Ignorance of the system parameters, while directly related to the system structure, is due to the inherent modeling error present in even the best structural analysis computer codes and to the limitation of testing huge, lightweight LSS on earth; this produces constant but poorly known system parameters. There is a very clear need for an adaptive LSS control methodology that can begin with the best available computed parameters and self-tune its way toward the correct parameters while stably controlling the LSS and, possibly, adapting to variable configurations.

A great variety of adaptive control schemes exists for lumped parameter, small scale systems [11]; in particular, model reference adaptive methods have achieved a great amount of success in producing stable, convergent adaptive controllers, and adaptive observers for systems whose structure is known and whose parameters are constant but poorly known or slowly time-varying, e.g., [12]-[25]. Adaptive schemes may be direct, i.e., the available control parameters are directly adjusted (adapted) to improve the overall system performance, e.g., [25]-[26], or indirect, i.e., the system parameters are identified (based on the assumed system structure) and the control commands are generated from these parameter estimates as though they were the actual values, e.g., [20], [24], [27].

The abundance of adaptive control methods is overwhelming and an understanding of the interrelationships and structural commonality of these methods is desperately needed; see, e.g., [28], [29], for some beginnings in this direction. Furthermore, the use of such methods on distributed parameter or large scale systems, like LSS, is greatly limited by the ROM problem - the adaptive scheme must be based on a ROM of the actual system

and, hence, the order of the model is, and must remain, substantially lower than the controlled system. In addition, it seems essential that the LSS adaptive controller provide a stabilizing control in such highly oscillatory systems as LSS.

This paper develops a framework for LSS adaptive control problems and points out generic problems in the use of the most natural direct and indirect adaptive approaches. In other forms, these problems will haunt every use of adaptive control on LSS and must be solved before the valuable benefits of adaptive control can meet the needs of this new application area. A few preliminary attempts at adaptive control for specific distributed parameter systems or LSS have been made in [30]-[36]; also, for the corresponding parameter identification problem for distributed parameter systems, see [37].

## 2.0 NONADAPTIVE LSS CONTROL

Following [6], the LSS may be described by the partial differential equation:

$$m(x) u_{tt}(x,t) + D_0 u_t(x,t) + A_0 u(x,t) = F(x,t) \quad (2.1)$$

where  $u(x,t)$  represents (possibly, a vector of generalized) displacements of the structure  $\Omega$  off its equilibrium position due to transient disturbances and the applied force distribution  $F(x,t)$ . The mass distribution  $m(x)$  is positive and bounded on  $\Omega$ . The internal restoring forces of the structure are represented by  $A_0 u$  where  $A_0$  is an appropriate differential operator with domain  $D(A_0)$  defined in a Hilbert space  $H_0$  with inner product  $(\dots)_0$ . In most cases,  $A_0$  has discrete spectrum, i.e.,

$$A_0 \phi_k = \omega_k^2 \phi_k \quad (2.2)$$

where  $\omega_k$  are the mode frequencies of vibration and  $\phi_k(x)$  are the mode shapes. The damping term  $D_0 u_t$  is generated by an appropriate  $A_0$ -bounded differential operator and may represent gyroscopic damping as well as the very small ( $\sim 1\%$  critical) natural damping expected in the LSS.

The applied force distribution is given by

$$F(x,t) = F_C(x,t) + F_D(x,t) \quad (2.3)$$

where  $F_D$  represents external disturbances and  $F_C$  represents the control forces due to  $M$  actuators:

$$F_C(x,t) = B_0 f = \sum_{i=1}^M b_i(x) f_i(t) \quad (2.4)$$

where  $b_i$  are the actuator influence functions (usually point devices) and  $f_i$  are the control commands. Observations are produced by  $P$  sensors:

$$y = C_0 u + C_0' u_t \quad (2.5)$$

where  $y_j(t) = (c_j, u)_0 + (c_j', u_t)_0$  for  $1 \leq j \leq P$  with  $c_j$  being the position sensor influence functions and  $c_j'$  the velocity sensor ones (usually point devices).

The state variable form of (2.1) and (2.3)-(2.5) is obtained by taking

$$v(x,t) = [u(x,t), u_t(x,t)]^T$$

in  $H \equiv D(A_0^{\frac{1}{2}}) \times H_0$  with energy norm:

$$\|v\|^2 = (mu_t, u_t) + (A_0^{\frac{1}{2}} u, A_0^{\frac{1}{2}} u) \quad (2.6)$$

This produces

$$\begin{cases} v_t = Av + Bf; v(0) = v_0 \\ y = Cv \end{cases} \quad (2.7)$$

where we consider the disturbance-free case ( $F_D \equiv 0$ ) and define  $B \equiv [0 \ B_0]^T$ ,

$C \equiv [C_0 \ C_0']$  and  $A \equiv \begin{bmatrix} 0 & I \\ -A_0 & -D_0 \end{bmatrix}$ . This distributed parameter system is

very oscillatory in the sense that the semigroup  $U(t)$  generated by  $A$  has very little damping:

$$\|U(t)\| \leq M_0 e^{-\epsilon t} \text{ for } t \geq 0 \quad (2.8)$$

where  $\epsilon \geq 0$  and small and  $M_0 \geq 1$ .

The desired performance of the actively controlled LSS greatly effects the design of the controller. Many desirable properties of the active structure can be obtained with constant feedback gains applied to the system state  $v(x,t)$ ; such solutions arise for regulator problems and stabilization (pole placement) problems for LSS. However, the full (infinite

dimensional) state  $v$  is never available from a distributed parameter system; only the  $P$  sensor outputs  $y$  are available.

Implementable controllers for LSS (and most distributed parameter systems) must be based on finite dimensional on-board control computers which process the sensor outputs  $y$  and produce control commands  $f$ ; thus, a reduced-order model (ROM) of the system (2.7) must be used for the controller design. A ROM can be obtained by projecting the system (2.7) in  $H$  onto an appropriate finite dimensional subspace  $H_N$ ; the projections  $P$  (onto  $H_N$ ) and  $Q$  (onto the residual subspace) are usually, but not always, orthogonal. Let  $v_N = Pv$  and  $v_R = Qv$  and, from (2.7), we obtain:

$$\dot{v}_N = A_N v_N + A_{NR} v_R + B_N f \quad (2.9)$$

$$\dot{v}_R = A_{RN} v_N + A_R v_R + B_R f \quad (2.10)$$

$$y = C_N v_N + C_R v_R \quad (2.11)$$

where  $A_N = PAP$ ,  $A_{NR} = PAQ$ ,  $B_N = PB$ , etc. The terms  $B_R f$  and  $C_R v_R$  are called control and observation spillover; the terms  $A_{NR} v_R$  are called model error. The ROM for this system is given by (2.9) and (2.11) with  $A_{NR} = 0$  and  $C_R = 0$ :

$$\begin{cases} \dot{v}_N = A_N v_N + B_N f \\ y = C_N v_N; v_N(0) = Pv_0 \end{cases} \quad (2.12)$$

The ROM state  $v_N$  and the residual state  $v_R$  form the true system state  $v$  with total energy  $\|v\|^2$  given by:

$$\begin{cases} v = v_N + v_R \\ \|v\|^2 = \|v_N\|^2 + \|v_R\|^2 \text{ (if projection is orthogonal)} \end{cases}$$

All implementable controller designs based on any ROM must be evaluated in closed-loop with the actual LSS (2.7), and it is in this evaluation that the effects of model error and spillover due to the residuals become apparent.

If the actual mode shapes  $\phi_k$  are known, the modal ROM is a sensible choice:

$$H_N = \{sp \phi_1 \dots \phi_N\}$$

and the model error terms  $A_{NR}$  and  $A_{RN}$  become zero. Of course, any collection of modes could be used; usually, the most easily excited or critical

ones will be chosen. However, in many cases, the partial differential operator  $A$  is too complex to provide closed-form mode shapes. Instead finite element approximations of the mode shapes are computed (e.g., via NASTRAN), and these approximate mode shapes can be used to form the ROM for controller design; note that some model error is present when these approximations are used. Henceforth, we will assume the actual mode shapes are available to simplify the discussion but much of our analysis remains valid for approximate mode shapes and other types of ROM.

Modern modal control (MMC) for LSS, as developed in [38], uses the modal (or approximate modal) ROM and develops a controller consisting of a state estimator based on the ROM and a constant gain control law:

$$\begin{cases} \dot{\hat{v}}_N = A_N \hat{v}_N + B_N f + K_N(y - \hat{y}) \\ \hat{y} = C_N \hat{v}_N, \hat{v}_N(0) = 0 \end{cases} \quad (2.13)$$

and

$$f = G_N \hat{v}_N \quad (2.14)$$

This controller design requires the ROM  $(A_N, B_N, C_N)$  to be controllable and observable for the calculation of control and estimator gains  $G_N, K_N$ . These conditions, in modal terms, provide insight into the number and location of actuators and sensors. From [38],  $(A_N, B_N, C_N)$  is controllable and observable, when position sensors are used ( $C_0 = 0$ ), if and only if

- (1)  $\min(P, M) \geq \max$  mode frequency multiplicity in the ROM
- (2) each sub-block of  $B_N$  and  $C_N$  associated with a mode frequency  $\omega_N$  of multiplicity  $\alpha_N$  must have rank at least equal to  $\alpha_N$ .

Similar results hold for other types of sensors, e.g., velocity, acceleration, or mixtures of types [38], [39]. These results are easy to interpret in terms of the mode shapes, e.g., if no repeated frequencies exist, then the above result says that a single actuator and sensor, not necessarily collocated, will do the job as long as neither is located at any of the ROM mode shape zeros. Since LSS have many symmetries and rigid body modes, it is not often that a LSS control problem will have a

controllable observable ROM with only one pair of devices; this has consequences for the adaptive control problem to be discussed later.

Let the estimator error  $e_N \equiv \hat{v}_N - v_N$  be defined and, from eqs. (2.9)-(2.11) and (2.13)-(2.14), obtain

$$\dot{v}_N = (A_N + B_N G_N) v_N + B_N G_N e_N \quad (2.15)$$

$$\dot{e}_N = (A_N - K_N C_N) e_N + K_N C_R v_R \quad (2.16)$$

$$\dot{v}_R = B_R G_N v_N + B_R G_N e_N + A_R v_R \quad (2.17)$$

This shows the effect of spillover on the closed-loop system: even though  $A_N + B_N G_N$ ,  $A_N - K_N C_N$ , and  $A_R$  are stable, the closed-loop system need not be stable. When either control or observation spillover is absent ( $B_R = 0$  or  $C_R = 0$ ), then stability is assured; otherwise, spillover causes pole-shifting and can induce instabilities [38], [40].

Bounds on the destabilizing effect of observation and control spillover were produced in [38] and can be extended to the case where some model error and small nonlinearities are present [41]. Such bounds give an indication of how much spillover the closed-loop system can tolerate.

A variety of methods have been suggested to reduce spillover [6]. One obvious way would be to prefilter the sensor outputs, with a bandpass filter, to substantially reduce observation spillover. This alleviates the worst pole-shifting problem; bounds on system performance with control spillover alone can be found in [42]. Note that the post-filter of the controller outputs could do the same job by reducing control spillover; this interchange of filter and controller is possible due to linearity and time-invariance. The trade-off with this means of reducing spillover is that the prefilter introduces phase distortion which can have a destabilizing effect of its own. Therefore, a very high order filter may be required to keep the phase distortion acceptable; phase-locked-loop quadrature filters may be another solution [43]. Even in nonadaptive LSS control, the spillover and model error problem is a fundamental one.

Finally, we should note in this section that digitally implemented controls would be based on discrete-time versions of the distributed parameter system (2.7). One such version is obtained by using a uniform time step  $\Delta t$ :

$$\begin{cases} v(k+1) = \Phi v(k) + E f_k \\ y(k) = C v(k) \end{cases} \quad (2.18)$$

where  $\Phi \equiv U(\Delta t)$  and  $E \equiv E_0 B = \int_0^{\Delta t} U(\tau) d\tau B$  and the control command is

a constant  $f_k$  over the interval  $(k-1)\Delta t \leq t < k\Delta t$ . Other versions of this could be obtained with nonuniform time steps. When the ROM procedure of projecting onto the subspace  $H_N$  is used, we obtain the discrete-time versions of (2.9)-(2.11):

$$v_N(k+1) = \Phi_N v_N(k) + \Phi_{NR} v_R(k) + E_N f_k \quad (2.19)$$

$$v_R(k+1) = \Phi_{RN} v_N(k) + \Phi_R v_R(k) + E_N f_k \quad (2.20)$$

$$y(k) = C_N v_N(k) + C_R v_R(k) \quad (2.21)$$

When  $H_N$  is the modal subspace, the above become:

$$v_N(k+1) = \Phi_N v_N(k) + E_0 B_N f_k \quad (2.22)$$

$$v_R(k+1) = \Phi_R v_R(k) + E_0 B_R f_k \quad (2.23)$$

$$y(k) = C_N v_N(k) + C_R v_R(k) \quad (2.24)$$

where (2.22) is the same as that obtained by directly discretizing the ROM in (2.12); if the exact mode shapes are not available, these two discretizations may yield different results. In addition, the sampling process can alias residual modes and increase observation spillover and the zero-order hold process can spread-out the control command signal spectrum and, hence, increase control spillover by increasing the energy in the residual mode spectrum; this has been observed and investigated in [44]. Therefore, the time discretization is a very important factor in the design of implementable LSS controllers.

### 3.0 TOWARD ADAPTIVE CONTROL OF LSS

In order to design MMC, or other controllers, for LSS as proposed in the previous section, we must have knowledge of the ROM parameters ( $A_N$ ,  $B_N$ ,  $C_N$ ). These parameters are obtained from modal data; they are the mode frequencies for  $A_N$  and the mode shapes at actuator and sensor locations for  $B_N$  and  $C_N$ , respectively. This data is required for three reasons:

- (1) to determine controllability and observability of the ROM and, hence, to help locate control devices effectively;
- (2) to design control and estimator gains;
- (3) to use in the state estimator's internal model.

However, we have noted in Sec. 1.0 the sources of error for this data; consequently, a need arises for an adaptive version of the MMC of Sec. 2.0.

The most logical and reasonable procedure to obtain adaptive controllers for a LSS seems to be the following:

#### Procedure for Adaptive LSS Control

- (a) choose a "nice" reduced-order model (ROM); a modal ROM would be the obvious choice;
- (b) use your "favorite" lumped parameter adaptive control scheme;
- (c) design the adaptive controller as though the ROM were the actual LSS to be controlled, i.e., ignore the unmodeled residual part of the structure;
- (d) use this adaptive controller in closed-loop with the actual LSS and hope for the best.

There is nothing wrong with following this procedure as a best first guess - in a way, there is little else that one can do to produce an implementable adaptive LSS controller.

In some cases, spillover is sufficiently small or enough other mathematical structure is present in the system, e.g., a high level of damping in the distributed parameter system [31] or low level of performance required from the controller (increased damping via direct velocity feedback) [32], to allow the adaptive controller to operate. However, these situations are rare with LSS and one would not like to count on the "generosity of nature" or the temporary suspension of Murphy's Law as part of the above design procedure. Consequently, we would add the following items to that procedure:

- (e) analyze computer simulations of higher-order models of the LSS in closed-loop with the adaptive controller based on the lower-order ROM (e.g., simulate more modes than you plan to control);



- (f) investigate the specific mathematical mechanisms whereby the residual (unmodeled) part of the LSS couples into a given adaptive control scheme (e.g., find out where and how spillover affects the adaptive controller);
- (g) obtain mathematical results on the amount of spillover and/or model error that can be tolerated in the closed-loop system and still provide adequate adaptation and control;
- (h) develop spillover and model error compensation schemes to augment the adaptive controller when the residuals cannot be tolerated (e.g., when the conditions of (g) are not satisfied);
- (i) recheck (g) with this compensation in the closed-loop system.

We believe that, within the basic framework of LSS control as described in Sec. 2.0, this Augmented Procedure (a)-(i) will go a long way toward revealing the problems of adaptive LSS control (and indeed, most other adaptive control situations where control must be based on a ROM) and will help to focus needed attention on these crucial issues. For example, although Step (e) would be done most likely at some point in the system development phase of any project as complex as the construction and operation of a LSS, often it is done much too late and the design is "set in concrete (or in this case, graphite-epoxy)"; the other steps (f)-(i) may not be done at all. Yet, ignoring the effects of the residual unmodeled LSS can produce some disastrous behavior in the adaptive controller; this was pointed out quite clearly in the LSS example in [36].

When an adaptive scheme is applied to such a lightly damped, oscillatory system as a LSS, the stability of the closed-loop system during adaptation is a necessity; therefore, we do not view convergence and (global) stability results as luxuries for adaptive LSS control and shall only consider appropriate those lumped parameter adaptive schemes for which such results are available. However, even a globally stable adaptive scheme may prove to be unstable when it is used in closed-loop with the actual LSS instead of the ROM on which it was based. This is not a failure of the adaptive scheme; it is a failure to satisfy the mathematical hypothesis of the stability result associated with the scheme.

In the rest of this section, we shall study the use of two very general adaptive schemes which seem to illustrate the problems and potential of adaptive LSS control: the indirect schemes of [23]-[24], which use an adaptive observer and operate in continuous time, and the discrete-time, direct or indirect, schemes which are based on an Autoregressive Moving Average (ARMA) model of the controlled plant, e.g., [25]-[27]. These approaches represent a good cross-section of available lumped parameter adaptive control schemes which have been shown to possess the desired stability properties. We emphasize that the point of this section is not to criticize or slander these schemes; rather, we mean to point out where the hypotheses of their stability results are violated, and must be modified, when we attempt to use them on LSS. We feel that consideration of these approaches within the context of the Augmented Procedure for Adaptive LSS Control (a)-(i) will illustrate the generic difficulties in the application of existing, well-behaved, lumped parameter adaptive control schemes to LSS.

### 3.1 Multivariable Systems Converted to Scalar Systems

The results of many stable adaptive schemes, e.g., [13]-[14], [23]-[24], are limited to a single actuator and/or a single sensor; yet, we have seen in Sec. 2.0 that most LSS control problems will involve multiple actuators and sensors. One way to deal with this (although, admittedly it has its drawbacks) is to convert the controllable observable LSS problem via output feedback into one that is controllable and observable from a single actuator and/or single sensor; this can be done with almost any output feedback gains [45]-[47]. These gains would have to be based on the best available calculated ROM data and the designer must hope that they will continue to do their job during adaptation.

The output feedback modifies the original system (2.9)-(2.11) to become:

$$\dot{v}_N = (A_N + B_N H_N C_N) v_N + (A_{NR} + B_N H_N C_R) v_R + b_N f \quad (3.1)$$

$$\dot{v}_R = (A_{RN} + B_R H_N C_N) v_N + (A_R + B_R H_N C_R) v_R + b_R f \quad (3.2)$$

$$y = c_N^T v_N + c_R^T v_R \quad (3.3)$$

where  $H_N$  is the output feedback gain matrix,  $b_N$ ,  $b_R$ ,  $c_N$ ,  $c_R$  are vectors,

and  $f, y$  have been renamed. Let  $A_N + B_N H_N C_N$  be  $\bar{A}_N$ , etc. and we have that the new ROM  $(\bar{A}_N, b_N, c_N^T)$  is a controllable, observable single-input, single-output system and (3.1) and (3.2) become:

$$\dot{v}_N = \bar{A}_N v_N + \bar{A}_{NR} v_R + b_N f \quad (3.4)$$

$$\dot{v}_R = \bar{A}_{RN} v_N + \bar{A}_R v_R + b_R f \quad (3.5)$$

### 3.2 Indirect Adaptive Controller Design

We apply the design of the adaptive controller in [24] directly to the ROM consisting of (3.4) and (3.3) with the assumption, for now, that

$$\bar{A}_{NR} = 0 \text{ and } c_R^T = 0.$$

The control law is given by

$$f(t) = g_N^T \hat{v}_N(t) + f_c(t) \quad (3.6)$$

where  $g_N$  is a constant gain vector,  $f_c$  is a "sufficiently rich" external signal (more about this later), and  $\hat{v}_N$  is derived from the following adaptive observer (or state estimator):

$$\dot{\hat{v}}_N = F \hat{v}_N + g y + h f; \hat{v}_N(0) = \hat{v}_0 \quad (3.7)$$

where  $F$  is an arbitrary, stable matrix and  $g, h$  are unknown parameter vectors. The appropriate matching conditions are:

$$\begin{cases} F + G^* c_N^T = \bar{A}_N \\ h^* = b_N \\ \hat{v}_0 = \hat{v}_N(0) \end{cases} \quad (3.8)$$

where  $g^*, h^*$  are constant.

Let  $p_0^* \equiv [g^{*T} h^{*T} v_N(0)^T]^T$  and note that

$$\dot{p}^* = \tilde{F} p^*; p^*(0) = p_0 \quad (3.9)$$

where  $\tilde{F} \equiv \text{diag} [0 \ 0 \ F]$  and, when  $g = g^*, h = h^*$ , we have from (3.8) and (3.7):

$$v_N(t) = [M(t) \ I_N] p^*(t) \quad (3.10)$$

where  $M(t) \equiv \int_0^t e^{F(t-\tau)} [I_N y(\tau) \ I_N f(\tau)] d\tau$  and we have used the fact that  $h, f$  are scalars.

Now (3.7) can be rewritten as

$$\hat{v}_N(t) = [M(t) \ I_N] p(t) \quad (3.11)$$

$$\dot{M}(t) = F M(t) + [I_N y(t) \ I_N f(t)] \quad (3.12)$$

$$M(0) = 0 \quad (3.13)$$

where  $M(t)$  is as defined in (3.10), and we have yet to produce the adaptive law to generate  $p(t)$ . This adaptive law is given by the following:

$$\dot{p}(t) = \tilde{F} p(t) - \alpha(t) [R(t) p(t) + r(t)] \quad (3.14)$$

where  $p(0) = p_0$  is arbitrary and

$$\begin{cases} \dot{R}(t) = -q R(t) - \tilde{F}^T R(t) - R(t) \tilde{F} + [M(t) \ I_N]^T c_N c_N^T [M(t) \ I_N] \\ R(0) = 0 \end{cases} \quad (3.15)$$

and

$$\begin{cases} \dot{r}(t) = -q r(t) - \tilde{F}^T r(t) - [M(t) \ I_N]^T c_N y(t) \\ r(0) = 0 \end{cases} \quad (3.16)$$

where  $\alpha(t)$  is the adaptive gain and the constant  $q$  is chosen to exceed twice the absolute value of the real parts of the eigenvalues of  $F$ . The adaptive gain is chosen so that

$$\alpha(t) = \gamma + \|\dot{\mu}(t)\| \quad (3.17)$$

where

$$\dot{\mu}(t) = -\lambda \mu(t) + (N/2)^{1/2} (\|y(t)\| + \|f(t)\|) \quad (3.18)$$

with  $\lambda$  positive and  $F + F^T \leq 2\lambda I_N$ .

### 3.3 Convergence Results: What Goes Wrong?

All of the above is exactly as stated in [24] where it is also shown in Appendix I and II that

$$\mu(t) \geq \|M(t)\| \quad (3.19)$$

and, with  $f(t)$  sufficiently rich in frequencies, there is a  $t_1$  such that

$$R(t) \geq \rho I_N > 0 \text{ for all } t \geq t_1 \quad (3.20)$$

In addition, it is shown in Appendix III that

$$R(t) p^*(t) + r(t) \equiv 0 \quad (3.21)$$

This is very crucial to the stability results of [24] and it is here that observation spillover (i.e., the fact that we are using a ROM) appears - (3.21) is not valid when  $c_R \neq 0$ ; however,

$$R(t) p^*(t) + r(t) = \Delta_R(t) \quad (3.22)$$

$$\text{where } \Delta_R(t) \equiv \int_0^t e^{-(q + \tilde{F}^T)(t - \tau)} [M(\tau) I_N]^T c_N c_R^T v_R(\tau) d\tau$$

Let  $e_N(t) \equiv \hat{v}_N(t) - v_N(t)$  and  $\Delta p(t) \equiv p(t) - p^*(t)$ ; then

$$e_N(t) = [M(t) I_N] \Delta p(t) \quad (3.23)$$

and

$$\Delta \dot{p}(t) = [\tilde{F} - \alpha(t) R(t)] \Delta p(t) - \alpha(t) \Delta_R(t) \quad (3.24)$$

Consider  $V(t) \equiv \Delta p(t)^T \Delta p(t)$  and we obtain:

$$\dot{V}(t) \leq -\gamma p V(t) - 2\gamma \Delta_R(t) \Delta p(t) \quad (3.25)$$

where  $V(0) = \Delta p_0^T \Delta p_0$  and  $\Delta p_0 = p_0 - p_0^*$ .

This follows [24] except for the additional term in (3.25); also note that

$$\|e_N(t)\|^2 \leq V(t) [1 + \mu(t)] \quad (3.26)$$

Let  $\Delta v_N(t) \equiv v_N(t) - v_N^*(t)$  and  $\Delta v_R(t) \equiv v_R(t) - v_R^*(t)$  where  $v_N^*$  and  $v_R^*$  represent the ideal states of (3.4)-(3.5) when the parameters are exactly known:

$$\dot{v}_N^* = (\bar{A}_N + b_N g_N^T) v_N^* + \bar{A}_{NR} v_R^* + b_N f_c \quad (3.27)$$

$$\dot{v}_R^* = (\bar{A}_{RN} + b_N g_N^T) v_N^* + \bar{A}_R v_R^* + b_R f_c \quad (3.28)$$

with  $v_N^*(0) = v_N(0)$  and  $v_R^*(0) = v_R(0)$ .

When the implementable adaptive control law (3.6) is used, we obtain:

$$\Delta \dot{v}_N = (\bar{A}_N + b_N g_N^T) \Delta v_N + \bar{A}_{NR} \Delta v_R + b_N g_N^T e_N \quad (3.29)$$

$$\Delta \dot{v}_R = (\bar{A}_{RN} + b_N g_N^T) \Delta v_R + \bar{A}_R \Delta v_R + b_R g_N^T e_N \quad (3.30)$$

and we have the following result:

**THEOREM 3.1:** Assume

- (1)  $c_R = 0$
- (2)  $\bar{A}_N + b_N g_N^T$  stable
- (3)  $\bar{A}_R$  stable
- (4)  $\bar{A}_{NR} = 0$
- (5)  $f_c(t)$  sufficiently rich (i.e., it has at least  $3N/2$  distinct frequencies)

Then there is a  $\delta > 0$  such that for all  $\|\Delta p_0\|^2 \leq \delta$ :

- (a)  $e_N(t)$  is bounded and (eventually) vanishes with an arbitrary exponential rate
- (b)  $\lim_{t \rightarrow \infty} \Delta v_N(t) = 0$
- (c)  $\lim_{t \rightarrow \infty} \Delta v_R(t) = 0$ .

Therefore, even though the closed-loop system with the adaptive controller is highly nonlinear, it is stable while the adaptation is taking place. In particular, (1), (3), and (4) are satisfied if there is no observation spillover ( $c_R = 0$ ) and no model error ( $\bar{A}_{NR} = 0$ ) and some damping in the residuals (i.e.,  $\bar{A}_R$  is stable).

Proof: This result follows from the results of [24] because  $\Delta_R(t) \equiv 0$  in (3.22) when  $c_R^T = 0$ . The stability of (3.29)-(3.30) is determined by  $\bar{A}_N + b_N g_N^T$  and  $\bar{A}_R$  when  $\bar{A}_{NR} = 0$ . Also, if  $c_R = 0$ , then  $\bar{A}_{NR} = A_{NR}$ ,  $\bar{A}_R = A_R$ , and  $c_R^T = 0$  since it is a row vector of  $C_R$ . #

Note that the stability of the system with adaptive control is determined by that of (3.27)-(3.28) - the ideal case where the parameters are known and the external signal  $f_c$  is applied. This is natural, since adaptation cannot take place without  $f_c$  present; however, after adaptation, we would most likely want to turn off  $f_c$ . In addition, we could choose the  $f_c$  signal so as not to excite the residual frequencies whenever sufficient spectral separation is present. Still, we would need to turn on  $f_c$  now and then, in order to "tune-up" the controller.

### 3.4 Spillover Compensation for the Indirect Adaptive Controller

The above result is merely a slight extension of the results of [24] to a special case of the adaptive controller based on an ROM instead of

the full system. However, it does suggest that some form of compensation should be used to eliminate the observation spillover. Such compensation must be essentially independent of the parameters of the ROM; yet, most methods of spillover reduction require knowledge of the ROM (and some residual) parameters.

One approach to spillover compensation already suggested in Sec. 2.0 is prefiltering the sensor outputs to remove or greatly reduce the term  $C_R v_R(t)$ . Such prefiltering can be achieved with low-pass or band-pass filters when the ROM frequencies are known and separated from the residual frequencies. However, the modal frequency data is part of the poorly known parameter data.

In an attempt to resolve this predicament, we could try using phase-locked loop (PLL)-based filters with the center frequency of each loop tuned to the best approximation available of the corresponding ROM data. The PLL will adapt itself until it tracks the actual mode frequency, and after "lock-on," it will behave as a narrow band-pass, linear filter which tunes out the observation spillover from the other frequencies [43]. Of course, sufficient spectral separation must be present, the calculated values of the ROM modal frequencies must be sufficiently good, the distortion introduced by the filter must be sufficiently small, and the adaptive controller must not shift the poles around too much. Thus, the PLL prefilter is not a panacea! But, it might work to reduce spillover and, if it does, it has the added advantage that its output could also reveal better estimates of the modal frequencies; this would take some of the load off the adaptive observer. If the adaptation mechanism causes too much pole shifting, the ROM frequencies could be excited via  $f_c$  and identified in open-loop by the PLL filters before the adaptive controller is turned on. Note that modal frequency data is usually better known, via computer approximation, than modal shape data; hence, this approach might not be unreasonable. Another possible, but untried, approach to spillover compensation might be an adaptive version of the orthogonal filter in [48].

Note that some prefiltering (and postfiltering) always takes place due to the bandwidth limitations of the sensors (and actuators). Whether this can be used to advantage in adaptive LSS control is still a matter for speculation, e.g., [33].

A problem that arises with the use of the results in [24] and their modification to LSS is that the constant feedback gains  $g_N^T$  must be calculated, in advance, to stabilize  $\bar{A}_N + b_N g_N^T$ . It would be better if these gains were adapted along with the parameters in the observer. Of course, after adaptation has taken place, they could be recalculated from the "tuned-up" parameters, but, in some cases, the adaptation phase is never over, e.g., slowly-varying parameters. Other approaches could be used for adaptive pole-placement, e.g., [20], [21], but these also have their limitations.

### 3.5 ARMA-Gettin'

Many discrete-time adaptive control schemes depend on an Auto-Regressive Moving Average (ARMA) representation of the plant in discrete-time, e.g., [25]-[27]:

$$y(k + N) = \sum_{r=1}^N \alpha_r y(k + r - 1) + \sum_{r=1}^N \beta_r f(k + r - 1) \quad (3.31)$$

for some  $N$  and appropriate matrices  $\alpha_r$ ,  $\beta_r$ . What the ARMA says is that, after  $N$  time steps, the present output is only related to the past  $N$  outputs and inputs. Existence of an ARMA is directly related to the finite dimensionality of the plant ( $N$  is usually that dimension) and is obtained using the Cayley-Hamilton theorem for matrices. For LSS, only a quasi-ARMA can exist; these were considered in detail in [49]. From the Appendix of [49], we obtain the quasi-ARMA for the LSS (2.18) or (2.19)-(2.21):

$$y(k + N) = \sum_{r=1}^N \alpha_r y(k + r - 1) + \sum_{r=1}^N \Gamma_r E_N f(k + r - 1) + R(k) \quad (3.32)$$

where

$$R(k) = C_R v_R(k + N) + \sum_{r=1}^N \Delta_r v_R(k + r - 1)$$

$$\Delta_r = \Gamma_r \Phi_{NR} - \alpha_r C_R$$



and  $\Gamma_r$  is defined in the Appendix of [49]. Since  $R(k) = 0$  when  $C_R = 0$  and  $\Delta_r = 0$ , we have the following result:

**THEOREM 3.2:** When the observation spillover ( $C_R$ ) and the model error term ( $\phi_{NR}$ ) are both zero, the quasi-ARMA (3.32) is a true ARMA for the LSS (2.18), and any stable adaptive scheme based on this ARMA will be globally stable when used in closed-loop with the actual LSS (2.18).

When the rather stringent hypothesis of Theo. 3.2 is not satisfied (as it may not be in practice), any adaptive LSS control scheme based on the quasi-ARMA (3.31) must ignore  $R(k)$  in order to be implementable. However,  $R(k)$  is the term where the residual effects - spillover and model error - enter the scheme and can cause instability. Again, as in Sec. 3.4, prefiltering or other compensation might be tried in an attempt to reduce or eliminate this term.

#### 4.0 CONCLUSIONS

In an attempt to point out the crucial issues and generic problems associated with adaptive control of large aerospace structures (LSS), we have reviewed the framework for nonadaptive LSS control (Sec. 2.0) and, within this framework, have proposed a general procedure, based on reduced-order models (ROM) of the LSS, for obtaining and assessing the problems of adaptive LSS controllers (3.0). In addition, we have taken a closer look at the use of certain well-known, lumped parameter, stable adaptive control schemes in this procedure. Taking these schemes as representative of the basic ideas present in all lumped parameter adaptive control approaches, we have obtained corresponding LSS adaptive controllers and found the following generic problems associated with adaptive LSS control:

- (1) LSS are distributed parameter or large scale systems; therefore, the plant dimension is always larger than the dimension of the adaptive controller, which must be based on a ROM;

- (2) LSS control must often be done with more than one actuator and sensor; conversion of multivariable to scalar systems via output feedback introduces problems of its own (e.g., stability of the residuals);
- (3) LSS control problems are often non-minimum phase due to noncollocated actuators and sensors;
- (4) Interaction of the residuals with the adaptive controller may negate the stabilizing properties of the controller due to observation spillover; this interaction is much worse due to the nonlinear nature of adaptive control;
- (5) Methods of spillover compensation for LSS often require knowledge of the RCM (and some residual) parameters - the very data that are poorly known;
- (6) The adaptation mechanism may shift the closed-loop frequencies around; this counteracts the benefits of any prefiltering unless sufficient spectral separation is maintained;
- (7) Indirect adaptive controllers need sufficient excitation from an external signal  $f_c$ ; however, this signal may substantially excite the residuals.
- (8) Discrete-time adaptive controllers can only be based on quasi-ARMA rather than strict ARMA representations of the LSS; this may negate the stability properties of such a controller.

Stable adaptation is essential for such highly oscillatory systems as LSS, yet our preliminary stability results, Theos. 3.1 and 3.2, both require that observation spillover be somehow completely eliminated before it reaches the adaptive control logic; certainly, this is not an easy thing to do in general! Perhaps, global stability is too much to ask for LSS adaptive control because observation spillover will always be present to some degree in LSS control. However, it seems reasonable to hope for the development of spillover bounds to give some idea of regions of stability for the successful operation of LSS adaptive

control. Some comparison should be made between stable adaptive controllers based on ROM and stable, robust control schemes, e.g., [50], [51].

In closing, we would like to say that it is not our intent to present a gloomy picture for the application of adaptive control to LSS. In fact, the need for adaptive control in LSS is already becoming quite clear, and recognition of this need comes, for a change, at an appropriate time - before any LSS have been built and put into space. However, the development of adaptive control for LSS will not take place overnight and will not be done by one or two people. Consequently, what we have tried to stress here for the interested researcher are some of the fundamental problems that arise and the basic steps which need to be taken toward the goal of successful adaptive control of LSS. In the long run, we have high hopes for the success of this endeavor and we expect that adaptive control theory will profit by its association with large aerospace structures, as well.

#### REFERENCES

1. "A Forecast of Space Technology 1980-2000," NASA SP-387 (Jan. 1976).
2. Large Space Structures - Challenge of the Eighties, Special Section in AIAA Astronautics and Aeronautics, 24-59 (Oct. 1978).
3. Glaser, P., et al., First Steps to the Solar Power Satellite, IEEE Spectrum, 52-58 (May 1979).
4. Dahlgren, J., and S. Gunter, Pointing and Control Technology Needs for the Future Automated Space Systems, presented at AIAA Conf. on Large Space Platforms, Los Angeles, CA (Sept. 1978).
5. Herzberg, R., et al., Dynamics and Control of Large Satellites, AIAA Astronautics and Aeronautics, 35-39 (Oct. 1978).
6. Balas, M., Some Trends in Large Space Structure Control Theory: Fondest Hopes; Wildest Dreams, presented at JACC, Denver, CO (June 1979); to appear in revised form in IEEE Trans. Autom. Contr.
7. Meirovitch, L., and H. Öz, An Assessment of Methods for the Control of Large Space Structures, presented at JACC, Denver, CO (June 1979).

8. Gran, R., and Rossi, M., A Survey of the Large Structures Control Problem, presented at 18th IEEE CDC, Ft. Lauderdale, FL (Dec. 1979).
9. Likins, P., Attitude Control of Nonrigid Spacecraft, AIAA Astronautics and Aeronautics (May 1971).
10. Likins, P., The New Generation of Dynamic Interaction Problems, AAS Rocky Mtn. G & C Conf., Keystone, CO (Mar. 1978).
11. Asher, R., Andrisani, D., and Dorato, P., Bibliography on Adaptive Control Systems, Proc. IEEE 64, 1226-1240 (1976).
12. Narendra, K., and Kudva, P., Stable Adaptive Schemes for System Identification and Control, Parts I and II, IEEE Trans. Syst., Man and Cybernetics SMC-4, 542-560 (1974).
13. Narendra, K., and Valavani, L., Stable Adaptive Observers and Controllers, Proc. IEEE 64, 1198-1208 (1978).
14. Narendra, K., and Valavani, L., Stable Adaptive Controller Design - Direct Control, IEEE Trans. Autom. Control AC-23, 570-583 (1978).
15. Lindorff, D., and Carroll, R., Survey of Adaptive Control Using Lyapunov Design, Int'l. J. Control 18, 897-914 (1973).
16. Landau, I. D., Model Reference Adaptive Systems - A Survey - What is Possible and Why?, ASME J. Dyn. Systems, Measurement, and Control, 119-132 (1972).
17. Landau, I. D., A Survey of Model Reference Adaptive Techniques - Theory and Applications, Automatica 10, 353-379 (1974).
18. Carroll, R., and Lindorff, D., An Adaptive Observer for Single-Input Single-Output Linear Systems, IEEE Trans. Autom. Control AC-18, 428-435 (1973).
19. Nagan, S., and Carroll, R., An Adaptive Observer and Identifier with an Arbitrarily Fast Rate of Convergence, presented at IEEE CDC (1976).
20. Nagan, S., and Carroll, R., Adaptive Identification and Pole Placement for Stable Plants, Univ. of S. Carolina, Report No. EE-CACL-TR-77-1 (1977).
21. Kraft, L. G., Controllable Form State Variables Obtained from an Arbitrarily Fast Adaptive Observer, presented at IEEE CDC (1976).
22. Kreisselmeier, G., Adaptive Observers with Exponential Rate of Convergence, IEEE Trans. Autom. Control AC-22, 2-8 (1977).

23. Kreisselmeier, G., The Generation of Adaptive Law Structures for Globally Convergent Adaptive Observers, IEEE Trans. Autom. Control AC-24, 510-513 (1979).
24. Kreisselmeier, G., Algebraic Separation in Realizing A Linear State Feedback Control Law by Means of an Adaptive Observer, to appear IEEE Trans. Autom. Control.
25. Monopoli, R., Model Reference Adaptive Control with an Augmented Error Signal, IEEE Trans. Autom. Control AC-19, 474-484 (1974).
26. Johnson, Jr., C. R., and Tse, E., Adaptive Implementation of One-Step-Ahead Optimal Control via Input Matching, IEEE Trans. Autom. Control AC-23, 865-872 (1978).
27. Goodwin, G., Ramadge, P., and Caines, P., Discrete Time Stochastic Adaptive Control, to appear IEEE Trans. Autom. Control.
28. Johnson, Jr., C. R., Input Matching, Error Augmentation, Self-Tuning, and Output Error Identification: Algorithmic Similarities in Discrete Adaptive Model Following, presented at 18th IEEE CDC, Ft. Lauderdale, FL (Dec. 1979).
29. Landau, I. D., Model Reference Adaptive Control and Stochastic Self-Tuning Regulators - Towards Cross-Fertilization, presented at Workshop on Adaptive Control, Univ. of Illinois, Urbana-Champaign, IL (May 1979).
30. Amouroux, M., Babary, J.-P., and El Jai, A., Adaptive Open Loop Control for a Class of Distributed Parameter Systems, Automatica 14, 603-607 (1978).
31. Hamza, M., and Sheirah, M., A Self-Tuning Regulator for Distributed Parameter Systems, Automatica 14, 453-463 (1978).
32. Benhabib, R., Iwens, R., and Jackson, R., Active Vibration Control of a Flat Plate Using Model Reference Adaptive Techniques, presented at 2nd Symp. on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA (June 1979).
33. Potter, J., and Ginter, S., A New Concept in Adaptive Structural Control, presented at 2nd Symp. on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA (June 1979).
34. Montgomery, R., and Johnson, Jr., C. R., The Dual Momentum Control Device for Large Space Systems: An Example of Distributed System Adaptive Control, presented at 12th Asilomar Conf. on Circuits, Systems, and Computers, Pacific Grove, CA (Nov. 1978).

35. Johnson, Jr., C. R., and Montgomery, R., A Distributed System Adaptive Control Strategy, IEEE Trans. on Aerospace and Electronics AES 15, 601-612 (1979).
36. Johnson, Jr., C. R., On Adaptive Modal Control of Large Flexible Spacecraft, presented at AIAA G & C Conf., Boulder, CO (Aug. 1979); to appear in revised form in J. Guidance and Control (1980).
37. Goodson, R., and Polis, M. (eds.), "Identification of Parameters in Distributed Systems: A Symp. of the Am. Auto. Control Council," JACC, Austin, TX (June 1974).
38. Balas, M., Feedback Control of Flexible Systems, IEEE Trans. Autom. Control AC-23, 673-679 (1978).
39. Hughes, P., and Skelton, R., Stability, Controllability, and Observability of Linear Matrix-Second-Order Systems, presented at JACC, Denver, CO (June 1979).
40. Ginter, S., and Balas, M., Attitude Stabilization of Large Flexible Spacecraft, presented at AIAA G & C Conf., Palo Alto, CA (Aug. 1978).
41. Balas, M., Control of Flexible Structures in the Presence of Certain Nonlinear Disturbances, presented at JACC, Denver, CO (June 1979).
42. Balas, M., Modal Control of Certain Flexible Dynamic Systems, SIAM J. Contr. and Opt. 16, 450-462 (1978).
43. Gustafson, D., and Speyer, J., Linear Minimum Variance Filters Applied to Carrier Tracking, IEEE Trans. Autom. Control AC-21, 65-73 (1976).
44. Ginter, S., Attitude Control of Large Flexible Spacecraft, M.S. Thesis, Mass. Inst. of Tech., Cambridge, MA (Sept. 1978).
45. Davison, E., and Wang, S., Properties of Linear Time-Invariant Multivariable Systems Subject to Arbitrary Output and State Feedback, IEEE Trans. Autom. Control AC-18, 24-32 (1973).
46. Heymann, M., On the Input and Output Reduceability of Multivariable Linear Systems, IEEE Trans. Autom. Control AC-15, 563-569 (1970).
47. Ding, C., Brasch, F., and Pearson, J., On Multivariable Systems, IEEE Trans. Autom. Control AC-15, 96-97 (1970).
48. Skelton, R., and Likins, P., Orthogonal Filters for Model Error Compensation in the Control of Non-rigid Spacecraft, J. Guidance and Control 1, 41-49 (1978).

49. Balas, M., and Johnson, Jr., C. R., Adaptive Control of Distributed Parameter Systems: The Ultimate Reduced-Order Problem, presented at IEEE CDC, Ft. Lauderdale, FL (Dec. 1979).
50. Gutman, S., and Leitmann, G., Stabilizing Feedback Control for Dynamic Systems with Bounded Uncertainty, presented at IEEE CDC (1976).
51. Nikiforuk, P., Gupta, M ., and Tamura, K., The Design of a Signal Synthesis Model Reference Adaptive Control System for Linear Unknown Plants, ASME J. Dyn. Syst., Meas., and Control 99, 123-129 (1977).

## G. REDUCED ORDER ADAPTIVE CONTROLLER STUDIES

C. Richard Johnson, Jr.  
Department of Electrical Engineering  
Virginia Polytechnic Institute and State University  
Blacksburg, Virginia

Mark J. Balas  
Department of Electrical and Systems Engineering  
Rensselaer Polytechnic Institute  
Troy, New York

## ABSTRACT

The quantitative effects of spillover on reduced order adaptive control are examined via a test example.

## 1. INTRODUCTION

A major assumption in the various recent proofs (1) - (3) of lumped-parameter system (LPS) adaptive control algorithms is the adequacy of the order of the controller. The model of the plant used in specifying the order of the controller is required to match or exceed that of the plant. For example (1), for the model reference adaptive control of the plant

$$y(k) = \sum_{i=1}^n a_i y(k-i) + \sum_{j=1}^m b_j u(k-j) \quad (1)$$

in the absence of accurate a priori specification of the plant parameters  $a_i$  and  $b_j$  an adjustable control law of the form

$$u(k-1) = \hat{c}_1(k-1)r(k-1) + \sum_{j=2}^m \hat{c}_j(k-1)u(k-j) - \sum_{i=1}^n \hat{d}_i(k-1)y(k-i) \quad (2)$$

is required to insure the asymptotic convergence of  $y$  to the output  $s$  of the model

$$s(k) = gr(k-1) + \sum_{i=1}^p f_i s(k-i). \quad (3)$$

Note that in (2) the summation limits  $m$  and  $n$  must correspond exactly to the moving-average and autoregression lengths in (1), respectively. As shown in (5), the problem of (1) - (3) is solvable via a variety of indirect and direct adaptive control techniques that prove quite similar in this exact order case. However, in the reduced order case where the control input is provided by

$$u(k-1) = \hat{c}_1(k-1)r(k-1) + \sum_{j=2}^p \hat{c}_j(k-1)u(k-j) - \sum_{i=1}^q \hat{d}_i(k-1)y(k-i) \quad (4)$$

C. R. Johnson, Jr. is supported by NASA Grant NAG-I-7. M. J. Balas is supported by NASA Grant NAS 9-16053.

with  $m > p$  and  $n > q$  these similarities disappear along with the ability for the output error  $s - y$  to converge to zero. In this case it is expected that the results of indirect and direct adaptive control will be quite distinguishable, which counters the current trend of exclaiming their equivalence (7)-(10).

The use of a reduced-order adaptive controller, such as (4), can arise from a desire to reduce control complexity with a commensurate reduction in controller sensitivity (11) or from necessity in an attempt to use a finite dimensional controller on an infinite dimensional system (12). A recent interest (13) - (15) in developing adaptive controllers for flexible structures by application of existing LPS adaptive controller strategies to truncated expansion descriptions of the distributed parameter system (DPS) behavior of flexible structures has led to two qualitative descriptions of the misbehavior of reduced-order adaptive controllers. These interpretations of the additional difficulties facing reduced-order adaptive controllers, which are bypassed by exact-order adaptive controllers, are summarized in the next section. The last section formulates a test problem, which initializes attempts to quantify these qualitative insights. The most current results of this investigation will be compiled in the conference presentation.

## II. REDUCED-ORDER EFFECTS IN ADAPTIVE CONTROL

Though numerous strategies have been espoused (16) for extracting a low order model from a too complex system description, none currently seem fully applicable to the real-time, recursive requirements of on-line adaptive control algorithms. These reduced order modeling techniques seem to fall into two broad categories: (i) Extract those "modes" (or component-subsystems) from the full system description that are most influential in the performance of the model in its subsequent use. This strategy is followed, for example, in (17) and (18). (ii) Parameterize the reduced structure to provide the best prediction of the desired output. This latter strategy will produce a model the "modes" of which need not correspond to any of those of the full system as noted in (14). This latter strategy includes the model reference approach prominent in adaptive systems (19).

The presentation in (14) emphasizes the difficulty of reduced-order adaptive control by contrasting these two model reduction strategies. Consider the approximation of

$$d(\theta, k) = \sum_{j=1}^N w_j(k) \phi_j(\theta), \quad (5)$$



where  $d$  can serve as the deflection of a flexible structure,  $W_j$  the modal amplitudes, and  $\phi_j$  the mode shapes for DPS or with  $\phi_j \equiv 1$  as the output of a decoupled (partial fraction expansion) description of a LPS with  $W_j$  as the individual "modal" outputs (20). Assuming that the modal amplitudes obey a difference equation such as

$$W_j(k) = \sum_{i=1}^n [a_{ji} W_j(k-1) + b_{ji} F_j(k-1)], \quad (6)$$

where the  $F_j$  are the modal inputs and the  $a_{ji}$  and  $b_{ji}$  are the unknown parameters of interest, the first strategy for adaptive identification of (6) requires extraction of the appropriate  $W_j$  from (5) forming

$$d_R(\theta, k) = \sum_j^{Mof[1, N]} W_j(k) \phi_j(\theta), \quad (7)$$

which as noted in (12), (14) and (20) remains an open problem without knowledge of  $N$  and the  $\phi_j$ . The second strategy would estimate the  $W_j$  such that

$$\hat{d}(\theta, k) = \sum_j^{Mof[1, N]} \hat{W}_j(k) \phi_j(\theta) \quad (8)$$

closely approximated  $d$  in (5) in, e.g., a least-squares sense. The estimated modal amplitudes  $\hat{W}_j$  need not provide useful values for the estimation of the  $a_{ji}$  and  $b_{ji}$  in (6) with  $W_j$  replaced by  $\hat{W}_j$  as noted in (14). In (8) the  $\hat{W}_j$  chosen at each time instant to closely match  $d$  to  $\hat{d}$  may not even obey an underlying difference equation such as (6). The problem with closing the adaptive control loop is then intensified by the difficulty in identifying (6) and then in feeding back the  $\hat{W}_j$  instead of the  $W_j$  for modal control.

The difference in various identification techniques in the reduced-order LPS case can be emphasized by a simple example from (21). Consider identifying the stable, second-order plant

$$y(k) = 1.1314y(k-1) - 0.25y(k-2) + 0.05u(k-1) - 0.4u(k-2) \quad (9)$$

using a first order model and three common adaptive identification schemes: the equation error formulated LMS algorithm (22), the gradient based output error formulated Stearn's algorithm (SA) (23), and the stability based output error formulated SHARF (24). The algorithms are

$$\text{LMS: } \hat{y}(k) = \hat{a}(k)y(k-1) + \hat{b}(k)u(k-1) \quad (10)$$

$$\hat{a}(k+1) = \hat{a}(k) + u[y(k) - \hat{y}(k)]y(k-1) \quad (11)$$

$$\hat{b}(k+1) = \hat{b}(k) + \rho[y(k) - \hat{y}(k)]u(k-1) \quad (12)$$

$$\text{SA: } \hat{y}(k) = \hat{a}(k)\hat{y}(k-1) + \hat{b}(k)u(k-1) \quad (13)$$

$$\hat{a}(k+1) = \hat{a}(k) + u[y(k) - \hat{y}(k)]\lambda(k) \quad (14)$$

$$\lambda(k) = \hat{y}(k-1) + \hat{a}(k)\lambda(k-1) \quad (15)$$

$$\hat{b}(k+1) = \hat{b}(k) + \rho[y(k) - \hat{y}(k)]\gamma(k) \quad (16)$$

$$\gamma(k) = u(k-1) + \hat{a}(k)\gamma(k-1) \quad (17)$$

$$\text{SHARF: } \hat{y}(k) = \hat{a}(k)\hat{y}(k-1) + \hat{b}(k)u(k-1) \quad (18)$$

$$\hat{a}(k+1) = \hat{a}(k) + uv(k)\hat{y}(k-1) \quad (19)$$

$$\hat{b}(k+1) = \hat{b}(k) + \rho v(k)u(k-1) \quad (20)$$

$$v(k) = y(k) - \hat{y}(k) + \lambda[y(k-1) - \hat{y}(k-1)]. \quad (21)$$

If the  $y$  in (9) were generated by a first order model as are (10), (13) and (18), then each of the three schemes (if  $u$ ,  $\rho$ , and  $\lambda$  were chosen to assure stability) converge to the unique correct values given sufficiently rich  $u$ . However for the reduced-order identification tasks of (9) - (21) each scheme yields different mean convergence points as shown in Figure 1. The input  $u$  in this case is white and zero-mean. The contours in Figure 1 represent constant normalized mean square error

$$E\{e^2(k)\} = E\{[y(k) - \hat{a}y(k-1) - \hat{b}u(k-1)]^2\} \quad (22)$$

for various choices of  $\hat{a}$  and  $\hat{b}$ . Note that the gradient-based SA is susceptible to local convergence and for this problem the SA mean convergence point is dependent on the initial guesses for  $\hat{a}$  and  $\hat{b}$ . Though if any of these schemes is used for simultaneous identification and control the input would no longer be white, clearly each would have a quite different control performance. It becomes unclear as to which would adaptively provide the best reduced-order model.

An alternate interpretation of the "dual" ill-effects of reduced-order adaptive control has emerged, also from the flexible structure control problem (12) (13). For this viewpoint consider separation of a single-input, single-output system state-space description of the reduced-order model and the remainder (or residual) as

$$\begin{bmatrix} \dot{x}_N(k+1) \\ \dot{x}_R(k+1) \end{bmatrix} = \begin{bmatrix} A_N & A_{NR} \\ A_{RN} & A_R \end{bmatrix} \begin{bmatrix} x_N(k) \\ x_R(k) \end{bmatrix} + \begin{bmatrix} b_N \\ b_R \end{bmatrix} u(k) \quad (23)$$

$$y(k) = \begin{bmatrix} c_N & c_R \end{bmatrix} \begin{bmatrix} x_N(k) \\ x_R(k) \end{bmatrix} \quad (24)$$

From (23) - (24) the "spillover" of the residual modes into the observation of  $y(k)$  via  $cx_R(k)$  and the spillover of the control designed for the reduced model into the residual modes  $x_R(k+1)$  via  $b_R u(k)$  are clearly displayed. Also the possible coupling of the reduced and model modes via  $A_{NR}$  and  $A_{RN}$  is immediately apparent. Shown in (10) and (15) is the predictable fact that if  $A_{NR} = 0$  and  $c_R = 0$  then, assuming the residual modes are not unstable, any full-order-stable adaptive controller identifying  $A_N$ ,  $b_N$ , and  $c_N$  explicitly or implicitly from  $y$  and  $u$  alone would retain its stability. If  $A_{NR} = 0$  but  $c_R$  and  $b_R$  are nonzero then the degradation of reduced-order adaptive control has two sources. Unmodeled components in  $y$  via  $cx_R$  will generate an error that is indistinguishable from parameter error causing adaptation. The application of control  $u$  to  $x_R$  via  $b_R u$  will cause further excitation of the residual modes further perturbing the reduced modeling of  $y$ . Not only will the parameter estimates become incorrect but also the state estimates from an adaptive observer will be incorrect leading to an unpredictable "controlled" response.

These two problems of inappropriate parameter and state estimation are the same ones noted in the earlier interpretation with  $W_1$  rather than  $W_2$  extraction. The ability to obtain  $d_R$  corresponds to an effective zeroing of  $c_R$ . Since such ability does not appear forthcoming in general, the effects of spillover on reduced-order adaptive control via the second strategy noted at the beginning of this section must be evaluated. A test example follows which is designed to provide insight into this problem.

### III TEST EXAMPLE

Consider controlling the second-order plant with transfer function

$$\frac{Y(z)}{U(z)} = \frac{b_1}{z - a_1} + \frac{cb_2}{z - a_2} \quad (25)$$

and therefore difference equation description

$$\begin{aligned} y(k) &= (a_1 + a_2)y(k-1) - (a_1 a_2)y(k-2) \\ &\quad - (b_1 + cb_2)u(k-1) - (b_1 a_2 - cb_2 a_1)u(k-2) \end{aligned} \quad (26)$$

under the assumption of a first order model. Using the separation technique of (23) - (24), (26) can be rewritten as

$$\begin{bmatrix} x_1(k-1) \\ x_2(k-1) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k) \quad (27)$$

$$y(k) = \begin{bmatrix} 1 & c \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad (28)$$

where, from (23) - (24),  $A_R = a_1$ ,  $A_R = a_2$ ,  $A_{NR} = A_{R1} = 0$ ,  $b_R = b_1$ ,  $b_R = b_2$ ,  $c_R = 1$ , and  $c_R = c$ . Note that any proportion of  $b_R$  could be distributed to  $b_R$  and  $c_R$  such that  $b_R c_R = b_R c$ . Therefore increased  $c$  corresponds to increased spillover. Clearly when  $b_R c_R = 0$  a first order model of (25) would be exact. Clearly in such a case as noted earlier any stable adaptive control scheme would be successful. Also, equally apparent is the degradation of the controller, adaptive or not, as  $b_R c$  becomes nonzero and the residual mode  $x_2$  contributes significantly to  $y$ .

This problem will be used to address various issues including: (i) The classical "rule-of-thumb" for model reduction is based on the relative time constants of the various modes, with the very "fast" segments considered removable. In discrete time systems, for positive  $a_1$ , the residual mode is assumed neglectable if  $10 \ln(a_1) > \ln(a_2)$ . The verity of this rule for adaptive control will be tested. (ii) The first-order adaptive control of the full system will be compared to the nonadaptive reduced-order control constructed with the ability to control one mode or the other, whichever proves "dominant" for the particular objective. (iii) Indirect and direct adaptive controllers will be compared via average tracking error at convergence and average control effort expended. The three identifiers of (10) - (21) will be used for the indirect schemes with

$$\hat{f}_1(k) = \frac{d}{b(k+1)} \quad (29)$$

$$\hat{f}_2(k) = \frac{c - a(k+1)}{b(k+1)} \quad (30)$$

and

$$\hat{u}(k) = \hat{f}_1(k)r(k) + \hat{f}_2(k)y(k). \quad (31)$$

The common adaptive controller from (6)

$$\hat{f}_1(k) = \hat{f}_1(k-1) + pr(k-1)[dr(k-1) + cy(k-1) - y(k)] \quad (32)$$

$$\hat{f}_2(k) = \hat{f}_2(k-1) + uy(k-1)[dr(k-1) + cy(k-1) - y(k)] \quad (33)$$

with  $u$  and  $p$  appropriately constrained will parameterize (31) for the direct case.

Preliminary results are ambiguous. None of the simulated tests clearly support the universal usefulness of neglecting high frequency modes, the superiority of modal rejection or attempted "full" adaptive tracking, or the clear benefit of indirect or direct adaptive techniques. In other words, one approach is better in one example than another, which requires further investigation of reduced-order adaptive control. The mixed evidence will be interpreted in the conference presentation.

### IV REFERENCES

- Goodwin, G. C., Ramadge, P. J., and Caines, P. E., "Discrete Time Multivariable Adaptive Control," IEEE Trans. on Aut. Control, Vol. AC-25, 1980.
- Narendra, K. S., and Lin, Y.-H., "Stable Discrete Adaptive Control," IEEE Trans. on Aut. Control, Vol. AC-25, 1980.
- Kreisselmeier, G., "Adaptive Control Via Adaptive Observation and Asymptotic Feedback Matrix Synthesis," submitted to IEEE Trans. on Aut. Control.
- Egardt, B., "Stability Analysis of Discrete Time Adaptive Control Schemes," submitted to IEEE Trans. on Aut. Control.
- Morse, A. S., "Global Stability of Parameter-Adaptive Control Systems," submitted to IEEE Trans. on Aut. Control.
- Johnson, C. R., Jr., "Input Matching, Error Augmentation, Self-Tuning, and Output Error Identification: Algorithmic Similarities in Discrete Adaptive Model-Following," IEEE Trans. on Aut. Control, Vol. AC-25, no. 4, August 1980.
- Egardt, B., "Unification of Some Continuous-Time Adaptive Control Schemes," IEEE Trans. on Aut. Control, Vol. AC-24, no. 4, August 1979, pp. 588-592.
- Narendra, K. S., and S. S. Sastry, "Direct and Indirect Model Reference Adaptive Control," Automatica, Vol. 15, No. 6, November 1979, pp. 653-664.
- Jung, L. and Landau, I. D., "Model Reference Adaptive Systems and Self-Tuning Regulators - Some Connections," Proc. 7th IFAC Congress, Vol. 3, 1978, pp. 1973-1980.
- Landau, I. D., "Adaptive Controllers with Explicit and Implicit Reference Models and Stochastic Self-Tuning Regulators - Equivalence and Duality Aspects," Proc. 17th IEEE Conf. on Dec. and Control, San Diego, CA, January 1979, pp. 1390-1395.

11 Goodwin, G. C. and Ramadge, P. J., "Design of Restricted Complexity Adaptive Regulators," IEEE Trans. on Aut. Control, Vol. AC-24, No. 4, August 1979, pp. 584-588.

12 Balas, M. J. and Johnson, C. R., Jr., "Adaptive Control of Distributed Parameter Systems: The Ultimate Reduced Order Problem," Proc. 18th IEEE Conf. on Dec. and Control, Ft. Lauderdale, FL, December 1979, pp. 1013-1017.

13 Johnson, C. R., Jr. and Montgomery, R. C., "ADistributed System Adaptive Control Strategy," IEEE Trans. on Aerospace and Electr. Sys., Vol. AES-15, No. 3, September 1979, pp. 601-612.

14 Johnson, C. R., Jr., "On Adaptive Modal Control of Large Flexible Spacecraft," J. Guidance and Control, Vol. 3, No. 4, July-August 1980.

15 Balas, M. J. and Johnson, C. R., Jr., "Toward Adaptive Control of Large Space Structures," in Applications of Adaptive Control, eds. K. S. Narendra and R. V. Monopoli, Academic Press, New York, 1980.

16 Genesio, R. and Milanese, M., "A Note on the Derivation and Use of Reduced-Order Models," IEEE Trans. on Aut. Control, Vol. AC-21, No. 1, February 1976, pp. 118-122.

17 Skelton, R. E. and Gregory, C. D., "Measurement Feedback and Modal Reduction by Modal Cost Analysis," Proc. 1979 Jt. Aut. Cont. Conf., Denver, CO, 1979, pp. 211-219.

18 Skelton, R. E., "Cost Decomposition of Linear Systems with Applications to Model Reduction," Int. J. Control, to appear.

19 Landau, Y. D., Adaptive Control: The Model Reference Approach, Marcel-Dekker, New York, 1979.

20 Johnson, C. R., Jr. and Balas, M.-J., "Distributed Parameter System Coupled ARMA Expansion Identification and Adaptive Parallel IIR Filtering: A Unified Problem Statement," Proc. 13th Asilomar Conf. on Circuits, Sys., and Computers, Pacific Grove, CA, November 1979.

21 Johnson, C. R., Jr. and Larimore, M. G., "Comments on and Additions to 'An Adaptive Recursive LMS Filter'," Proc. IEEE, Vol. 65, No. 9, September 1977, pp. 1399-1402.

22 Widrow, B., McCool, J. M., Larimore, M. G., and Johnson, C. R., Jr., "Stationary and Nonstationary Learning Characteristics of the LMS Adaptive Filter," Proc. IEEE, Vol. 64, No. 8, August 1976, pp. 1151-1162.

23 Parikh, D. and Ahmed, N., "On an Adaptive Algorithm for IIR Filters," Proc. IEEE, Vol. 66, No. 3, May 1978, pp. 585-588.

24 Larimore, M. G., Treichler, J. R., and Johnson, C. R., Jr., "SHARF: An Algorithm for Adapting IIR Digital Filters," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-28, No. 4, August 1980.

